DE

EVOLVENDARUM FUNCTIONUM

PRINCIPIIS AC FORMULIS

DISQUISITIONES NONNULLAE ANALYTICAE.

AUCTORE

LUDOVICO MARTINO LAUBER,

PHILOS. DOCT. ET AA. LL. MAG. IN GYMNASIO REGIO THORUNENSI MATHES ET PHYSIC. PROFESSORE.

THORUNI.

MDCCCXXVIII.

1828

BEROLINI, TYPIS G. REIMERIANIS.

PRAEMONENDA.

Inter gravissima subsidia, quae e calculo differentiali ad omnem Analysin et praesertim ad functionum theoriam redundant, primum profecto locum tenent ea, quae e functionibus fa, $f(a+i)$ in series evolutis emanant. Ex his enim evolutionibus desumuntur implicatiorum functionum $f(a+\alpha i+\beta i^2\ldots.)$, φfa evolvendarum rationes, atque evolutae hoc modo formulae omnia nobis suppeditant, quaecunque opus sunt ad quamvis functionem in seriem convertendam.

Ipsa autem functionum fa, $f(a+i)$ evolutiones tractandi ratio per se notatu quam dignissima est, tantamque habet vim ad illustrandas et confirmandas quas Mathesis sublimior novas atque egregias profert veritates, ut hanc disciplinam nec tam luculenter expositam, nec tam concinniter tractatam nunc spectares, nisi illa ratio ad amplam perspicuitatem et elegantiam provecta, novam quasi lucem in hac Matheseos parte accendisset.

Ex quo enim Illustrissimus Lagrange hanc ita comparavit tractandi rationem, ut notio de infinite parvo ante usitata, eaque quam maxime obscura et indeterminata, omnino e calculo exstirparetur, Analysis sublimior speciem vere paradoxam, quam hucusque mentita est plane abjecit, ejusque loco novam, eamque cum vero scientiae nomine convenientem formam acquisivit; de cujus sententiae veritate abunde nobis persuasum habebimus, si illa provocaverimus, quae Illustrissimus Euler de calculi differentialis definitione in initio praefationis operis celeberrimi, *Institutiones calculi differentialis* sponte fatetur:

„Quid sit calculus differentialis, atque in genere Analysis infinito-„rum? iis qui nulla adhuc ejus cognitione sunt imbuti vix explicari „potest, neque hic ut in aliis disciplinis fieri solet, exordium tractationis „a definitione commode sumere licet. Non quod hujus calculi nulla plane „detur definitio, sed quoniam ad eam intelligendam ejusmodi opus est „notionibus non solum in vita communi verum etiam in ipsa analysi „finitorum minus usitatis, quae demum in calculi differentialis pertracta-„tione evolvi atque explicari solent, quo fit, ut ejus definitio non antea „percipi queat, quam ejus principia jam satis dilucide fuerint perspecta."

Cum autem haec materia, neque ad systematicam aliquam condendam, neque ad faciliorem tantummodo et simpliciorem reddendam methodum tam exculta censenda sit, quin aliqua adhuc desiderentur; eaque de causa et propter rei ipsius, ubicunque de methodo excolenda agitur, gravitatem, quantum quisque sibi pro viribus huc conferendum sumserit, non omnino irritum fore sperare liceat: non alienum nobis visum est, quas invenimus novas hanc materiem tractandi methodos, et quas instituimus implicatiorum functionum evolutiones, quippe quae cum his rebus arctissime cohaerent, aliasque, quae huc spectant disquisitiones nonnullas, oblata nobis scribendi occasione publico nunc judicio subjicere.

De evolvendis functionibus fa, $f(a+i)$; disquisitio fundamentalis.

§. 1.

Functionum analyticarum duas proprie discernimus species, *distincte expressarum* et *formaliter designatarum* [algebraicarum et transcendentium], totidemque inde oriuntur valde inter se discrepantes formarum tractandarum rationes.

Etenim distinctarum functionum haec est indoles, ut quantitatem exhibendam per alias, easque inter se consueto et vulgari arithmetices algorithmo nexas quantitates luculenter repraesentent; quae igitur quantitatis exhibendae ad propositas exorta hoc modo relatio *arithmetica* commode vocanda erit. Functiones contra formaliter designatae nos monent quidem, adesse certam quandam quantitatis exhibendae ab una vel pluribus aliis propositis dependentiam; relatio autem illa arithmetica quae sit, plane ignoratur.

In illas igitur cadit, ita transformari, ut ad maximam, qua exhiberi possunt simplicitatem reducantur, quae reductio illis efficitur regulis, quae ad algorithmum arithmeticum et calculum potestatum et radicum spectant; in has autem, ita calculo subjici, ut inde ignota adhuc ralatio, eaque constante, si fieri potest, lege structa, nec non supputationi quam aptissima exeat: hoc autem fit, cum functio in seriem evolvitur, quae terminos secundum potestates integras unius e quantitatibus primariis (quarum scilicet expressio exhibenda functio est) progredientes teneat; et haec problemata praecipuum constituunt Analyseos sublimioris objectum.

Hoc autem loco omnes casus speciales uni generaliori subjicere poterimus, si formam maxime universalem fa, in qua vel secundariae quantitates ignorantur, in medium protulerimus; tunc autem sponte liquet, orituros nobis esse seriei coefficientes aeque generales, et quae a functione fa dependeant, eaque ipsa exprimantur.

§. 2.

Quaestio igitur nostra in hoc versatur, ut functionem generalem fa in seriem evolvamus, secundum potestates integras ipsius a progredientem; hoc est, ut coefficientes fictos aequationis:

$$fa = m + \alpha a + \beta a^2 \dots .$$

eliciamus.

Ex hac aequatione statim prodit $m = f\overset{o}{a}$, ubi ciffra supra a posita designat, hunc ipsius fa sumendum esse valorem, quem functio pro $a = o$ induit; ideoque habemus aequationem:

$$fa = f\overset{o}{a} + \alpha a + \beta a^2 \dots + \overset{n}{\alpha} a^n \dots \quad (A)$$

ex qua sequitur

$$\alpha = \overset{a=o}{\frac{fa - f\overset{o}{a}}{a}},$$

hoc est α aequatur huic valori quoti $\frac{fa - f\overset{o}{a}}{a}$, qui resultat, posito ubique $a = o$.

Habemus igitur primi coefficientis α valorem. Ceteri quidem quomodo enucleandi sint, nondum apparet; hoc tamen satis perspicitur, viam ad problema solvendum tali ratione muniendam esse, ut tam functionem fa quam seriem illi respondentem in directum computum trahamus. Hac enim ratione efficitur, ut nova nunc inter fictas et eas quantitates, a quibus illae dependent exeat relatio; et praesumere certe licet, jam e duplici hac relatione magis nobis perspicuum fore, quomodo coefficientes cum ipsa functione cohaereant, et ab ea derivandi sint.

§. 3.

Jam ad id, quod modo proposuimus efficiendum haecce methodus quasi sponte se offert:

„varietur quam simplicissime quantitas primaria, quod fit, si illa cum alia quanti-
„tate i pro lubitu sumpta per additionem combinatur."

Ponendo enim $(a+i)$ pro a, functio ipsa abit in $f(a+i)$ et series illi respondens formam

$$f\overset{o}{a} + \alpha(a+i) + \beta(a+i)^2 \dots + \overset{n}{\alpha}(a+i)^n \dots$$

induit; primo enim obtuitu hoc nobis apparet, expressionem $\overset{(a+i)=o}{f(a+i)}$, e nostra significandi methodo, idem esse quod $f\overset{o}{a}$. Habemus igitur aequationem

$$f(a+i) = f\overset{o}{a} + \alpha(a+i) + \beta(a+i)^2 + \gamma(a+i)^3 \dots + \overset{n}{\alpha}(a+i)^n \dots$$

vel hanc

$$\begin{aligned} f(a+i) = f\overset{o}{a} + \alpha a + \beta a^2 + \gamma a^3 \dots + \overset{n}{\alpha} . a^n \dots \\ + i(\alpha + 2\beta a + 3\gamma a^2 \dots + n\overset{n}{\alpha} . a^{n-1} \dots) \\ + i^2(\dots \dots \dots \dots) \\ \dots \dots \dots \dots \end{aligned}$$

indeque porro, ob $f\overset{o}{a} + \alpha a + \beta a^2 \dots = fa$

$$\overset{i=0}{\left\{\frac{f(a+i)-fa}{i}\right\}} = \alpha + 2\beta a \,\ldots\ldots + n\overset{n}{\alpha}.a^{n-1}\ldots\ldots$$

unde denique

$$\alpha = \overset{\substack{a=0\\ i=0}}{\left\{\frac{f(a+i)-fa}{i}\right\}}$$

sequitur.

Quod si nunc expressio $\overset{i=0}{\left\{\frac{f(a+i)-fa}{i}\right\}}$ brevitatis ergo per $f'a$ indicatur [quo nempe indice monemur, primum: quantitatem a functionis, cui index adfixus est, abire in $(a+i)$; tum: functionem ipsam a variata subtrahendam, quae resultant per i dividenda, et denique quantitatem $i=0$ ponendam esse. Eadem igitur indicis vi manente, expressio $f''a$ idem indicabit respectu $f'a$, quod $f'a$ respectu fa indicat, et ita porro], habemus:

$$\alpha = f'\overset{0}{a},$$

ideoque

$$f'a = f'\overset{0}{a} + 2\beta a + 3\gamma a^2 \ldots\ldots + n.\overset{n}{\alpha}a^{n-1}\ldots\ldots \quad (B).$$

Quae aequatio cum structura sua plane cum aequatione (A) §^{hi} 2. congruat, facile perspicitur, eandem methodum, quae ope aequationis (A) ad valorem coefficientis α eliciendum nos manuducebat, etiam hic ad coefficientis β valorem inveniendum simili successu usurpandam esse; indeque tertiam nobis orituram esse aequationem ad tertii coefficientis valorem eadem methodo determinandum et ipsam aptissimam; quae conclusiones cum quousque libuerit continuari possint, omnino nobis persuadetur, jam simplicissima illa substitutionis methodo viam ad problema solvendum patefactam esse.

§. 4.

Antequam autem hanc disquisitionem ulterius persequamur, demonstrandum erit, duos in §§. 2. et 3. pro α inventos valores identicos esse, ita ut emergat aequatio

$$\frac{\overset{a=0}{fa} - f\overset{0}{a}}{a} = \overset{\substack{a=0\\ i=0}}{\left\{\frac{f(a+i)-fa}{i}\right\}}.$$

Discerpatur utraque expressio in

$$\frac{\overset{\substack{a=0\\ i=0}}{f(a+i)}}{i} - \frac{\overset{\substack{a=0\\ i=0}}{fa}}{i}, \text{ et in } \frac{\overset{a=0}{fa}}{a} - \frac{\overset{a=0}{f\overset{0}{a}}}{a};$$

jam primo patet, valorem cujuspiam expressionis, quem post nonnullas substitutiones acquirit, invariatum manere, quomodocunque ordinem substitutionum perturbemus; erit igitur

$$\frac{f(a+i)}{i}\Bigg|^{a=0}_{i=0} = \frac{f(a+i)}{i}\Bigg|^{i=0}_{a=0}$$

expressio autem $\overset{a=0}{\dfrac{f(a+i)}{i}}$ statim reducitur in $\dfrac{fi}{i}$, et $\overset{i=0}{\dfrac{fi}{i}} = \overset{a=0}{\dfrac{fa}{a}}$; primi igitur harum expressionum termini per se aequales aestimandi sunt. Pari modo secundorum terminorum aequalitatem probare licet; est enim

$$\overset{a=0}{\frac{fa}{i}} = \frac{f\overset{o}{a}}{i}, \text{ et } \overset{i=0}{\frac{f\overset{o}{a}}{i}} = \overset{a=0}{\frac{f\overset{o}{a}}{a}}.$$

§. 5.

Resumamus nunc aequationem (B)

$$f'a = f'\overset{o}{a} + 2\beta.a + 3\gamma.a^2 + 4\delta.a^3 \ldots + n\overset{n}{\alpha}.a^{n-1} \ldots,$$

qua cum aequatione (A)

$$fa = f\overset{o}{a} + \alpha a + \beta a^2 + \gamma a^3 + \delta a^4 \ldots + \overset{n}{\alpha} a^n \ldots$$

comparata, statim colligere licet, singulos quosque coefficientes aequationis (B) eodem modo a functione $f'a$ derivandos esse, quo coefficiens aeque altus in aequatione (A) a functione fa derivatur.

Cum igitur sit in (A) $\alpha = f'\overset{o}{a}$, erit in (B) $2\beta = f''\overset{o}{a}$, ideoque in (A) $\beta = \dfrac{f''\overset{o}{a}}{1.2}$, unde seorsim in (B) $3\gamma = \dfrac{f'''\overset{o}{a}}{1.2}$, dehinc in (A) $\gamma = \dfrac{f'''\overset{o}{a}}{1.2.3}$, unde in (B) $4\delta = \dfrac{f^{\text{IV}}\overset{o}{a}}{1.2.3}$, vel in (A) $\delta = \dfrac{f^{\text{IV}}\overset{o}{a}}{1.2.3.4}$, et ita porro. Sit generaliter in (A) $\overset{n}{\alpha} = \dfrac{f^n\overset{o}{a}}{1.2.3\ldots n}$, erit ideo in (B) $(n+1)\overset{n+1}{\alpha} = \dfrac{f^{n+1}\overset{o}{a}}{1.2.3\ldots n}$, et dehinc in (A) $\overset{n+1}{\alpha} = \dfrac{f^{n+1}\overset{o}{a}}{1.2.3\ldots n.(n+1)}$; unde e notissima concludendi ratione colligimus, quam pro primis consecutivis coefficientibus invenimus legem eam universim valere, ideoque quaesitae seriei terminum generalem fore:

$$\frac{f^n\overset{o}{a}}{1.2\ldots n}.a^n.$$

§. 6.

Quam per solas ratiocinationes modo exhibuimus problematis solutionem, eam aeque facile per directum calculum, secundum eam, quae jam sub fine §[hi] 3. a nobis indicata est methodum, ad plenam certitudinem evehere valemus.

In

In aequatione enim

$$f'a = f'\overset{0}{a} + 2\beta . a + 3\gamma . a^2 + 4\delta . a^3 \ldots . + n\overset{n}{\alpha} . a^{n-1} \ldots .$$

substitutione $(a+i)$ pro a actu expedita, erit:

$$f'(a+i) = f'\overset{0}{a} + 2\beta(a+i) + 3\gamma(a+i)^2 + 4\delta(a+i)^3 \ldots . + n\overset{n}{\alpha}(a+i)^{n-1} \ldots .$$

vel

$$\begin{aligned} f'(a+i) = f'\overset{0}{a} + 2\beta a + 3\gamma a^2 + 4\delta a^3 \ldots . + n\overset{n}{\alpha} . a^{n-1} \ldots . \\ + i\{2\beta + 2.3.\gamma a + 3.4.\delta a^2 \ldots . + (n-1)n\overset{n}{\alpha} . a^{n-2} \ldots .\} \\ + i^2\{. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .\} \\ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{aligned}$$

dehinc

$$\overset{i=0}{\frac{f'(a+i) - f'a}{i}}, \text{ vel}$$

$$f''a = 2\beta + 2.3.\gamma a + 3.4.\delta a^2 \ldots . + (n-1)n\overset{n}{\alpha} . a^{n-2} \ldots .$$

unde

$$2\beta = f''\overset{0}{a}.$$

Sit nunc generaliter, quod jam ponere licet

$$f^m a = 1.2.3\ldots m\overset{m}{\alpha} + 2.3\ldots . m(m+1)\overset{m+1}{\alpha} . a + 3.4.5\ldots . (m+2)\overset{m+2}{\alpha} . a^2 \ldots .$$

ideoque

$$1.2.3\ldots . m\overset{m}{\alpha} = f^m . \overset{0}{a}$$

erit substitutione $(a+i)$ pro a instituta

$$\begin{aligned} f^m(a+i) &= 1.2.3\ldots m\overset{m}{\alpha} + 2.3\ldots(m+1)\overset{m+1}{\alpha}(a+i) + 3.4\ldots(m+2)\overset{m+2}{\alpha}(a+i)^2\ldots \\ &= 1.2.3\ldots m\overset{m}{\alpha} + 2.3\ldots(m+1)\overset{m+1}{\alpha} . a + 3.4\ldots(m+2)\overset{m+2}{\alpha} . a^2\ldots \\ &\quad + i\{2.3\ldots(m+1)\overset{m+1}{\alpha} + 2.3.4\ldots(m+2)\overset{m+2}{\alpha} . a\ldots\} \\ &\quad + i^2\{. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .\} \\ &\quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{aligned}$$

termini autem primae lineae valorem $f^m a$ complectuntur, habemus igitur

$$\overset{i=0}{\frac{f^m(a+i) - f^m a}{i}} = 2.3\ldots(m+1)\overset{m+1}{\alpha} + 2.3\ldots(m+2)\overset{m+2}{\alpha} . a\ldots$$

unde

$$2.3.4\ldots(m+1)\overset{m+1}{\alpha} = f^{m+1}\overset{0}{a},$$

quae formula cum priori

$$2.3.4\ldots . m\overset{m}{\alpha} = f^m\overset{0}{a}$$

comparata, dilucide regulam exhibet: singulos quosque coefficientes seriei $f\overset{0}{a} + \alpha a + \beta a^2\ldots$ e singulis quibusque praecedentibus orituros esse, numero accentuum characteri f affixorum de 1 augendo, et denominatore per hunc exortum numerum multiplicando. Cum ergo sit $\alpha = f'a$, erit

$$fa = f\overset{0}{a} + f'\overset{0}{a}.a + \frac{f''\overset{0}{a}}{1.2}a^2 \ldots + \frac{f^n\overset{0}{a}}{1.2\ldots n}.a^n \ldots$$

§. 7.

Functionem fa convenientissime excipit functio $f(a+i)$, quippe quae implicatiorum functionum simplicissima est; postulatur autem nunc, ut haec functio in seriem evolvatur secundum potestates integras alteriusutrius quantitatum i vel a pragredientem. Quae quaestio cum simillima sit praecedenti, jam praesumere licet, eandem qua in illo casu usi sumus methodum, etiam hic ex voto successuram.

Ponatur igitur

$$f(a+i) = fa + \alpha i + \beta i^2 + \gamma i^3 + \delta i^4 \ldots + \overset{n}{\alpha} i^n \ldots \quad (A)$$

unde

$$\alpha = \overset{i=0}{\frac{f(a+i)-fa}{i}} = f'a;$$

substituatur nunc $(i+k)$ pro i, et habebitur

$$\begin{aligned} f(a+i+k) &= fa + \alpha(i+k) + \beta(i+k)^2 + \gamma(i+k)^3 + \delta(i+k)^4 \ldots + \overset{n}{\alpha}(i+k)^n \ldots \\ &= fa + \alpha i + \beta i^2 + \gamma i^3 + \delta i^4 \ldots + \overset{n}{\alpha} i^n \ldots \\ &\quad + k\{\alpha + 2\beta.i + 3\gamma.i^2 + 4\delta.i^3 \ldots + n.\overset{n}{\alpha}.i^{n-1} \ldots\} \\ &\quad + k^2\{\ldots\ldots\ldots\} \\ &\quad \ldots\ldots\ldots \end{aligned}$$

dehinc

$$\overset{k=0}{\frac{f(a+i+k)-f(a+i)}{k}}, \text{ vel}$$

$$f'(a+i) = \alpha + 2\beta i + 3\gamma i^2 \ldots + n\overset{n}{\alpha}.i^{n-1} \ldots \quad (B)$$

unde

$$\alpha = f'(a+\overset{i=0}{i}) = f'a, \text{ et } 2\beta = \overset{i=0}{\frac{f'(a+i)-f'a}{i}} = f''a.$$

Quod si nunc aequationem (B) cum aequatione (A) comparamus, iisdem ut supra adhibitis conclusionibus colligere licebit, terminum generalem seriei (A) esse

$$\frac{f^n.a}{1.2\ldots n}.i^n;$$

quod autem ut directo calculo probemus, substituatur porro in aequatione (B) $(i+k)$ pro i, eritque

$$f'(a+i+k) = \alpha + 2\beta(i+k) + 3\gamma(i+k)^2 \ldots + n.\overset{n}{\alpha}(i+k)^{n+1} \ldots$$
$$= \alpha + \quad 2\beta i \quad + 3\gamma i^2 \ldots\ldots + n\overset{n}{\alpha} i^{n-1} \ldots$$
$$+ k\{2\beta + 2.3.\gamma i + 3.4.\delta i^2 \ldots + (n-1)n\overset{n}{\alpha}.i^{n-2} \ldots\}$$
$$+ k^2\{. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

dehinc

$$\overset{k=0}{\frac{f'(a+i+k)-f'(a+i)}{k}}, \text{ vel}$$

$$f''(a+i) = 2\beta + 2.3.\gamma i + 3.4.\delta i^2 \ldots + (n-1)n\overset{n}{\alpha}.i^{n-2} \ldots$$

unde

$$2\beta = f''a; \ 2.3.\gamma = \overset{i=0}{\frac{f''(a+i)-f''a}{i}} = f'''a.$$

Sit nunc generaliter

$$f^m(a+i) = 1.2\ldots m\overset{m}{\alpha} + 2.3\ldots(m+1)\overset{m+1}{\alpha}.i + 3.4\ldots(m+2)\overset{m+2}{\alpha}.i^2 \ldots$$

ideoque

$$1.2\ldots m\overset{m}{\alpha} = f^m a; \ 2.3\ldots(m+1)\overset{m+1}{\alpha} = \overset{i=0}{\frac{f^m(a+i)-f^m a}{i}} = f^{m+1}a,$$

erit substitutione $(i+k)$ pro i instituta

$$f^m(a+i+k) = 1.2\ldots m\overset{m}{\alpha} + 2.3\ldots(m+1)\overset{m+1}{\alpha}(i+k) + 3.4\ldots(m+2)\overset{m+2}{\alpha}(i+k)^2 \ldots$$
$$= 1.2\ldots m\overset{m}{\alpha} + 2.3\ldots(m+1)\overset{m+1}{\alpha}. \ i \ + 3.4\ldots(m+2)\overset{m+2}{\alpha}.i^2 \ldots$$
$$+ k\{2.3\ldots(m+1)\overset{m+1}{\alpha} + 2.3.4\ldots(m+2)\overset{m+2}{\alpha} i \ldots\}$$
$$+ k^2\{. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Cum igitur termini primae lineae valorem functionis $f^m(a+i)$ repraesentent, erit

$$\overset{k=0}{\frac{f^m(a+i+k)-f^m(a+i)}{k}} = 2.3\ldots(m+1)\overset{m+1}{\alpha} + 2.3\ldots(m+2)\overset{m+2}{\alpha}.i \ldots$$

unde

$$2.3\ldots(m+1)\overset{m+1}{\alpha} = f^{m+1}a, \text{ et } 2.3\ldots(m+2)\overset{m+2}{\alpha} = \overset{i=0}{\frac{f^{m+1}(a+i)-f^{m+1}a}{i}} = f^{m+2}a;$$

dehinc, iisdem ut in fine §hi praecedentis in opem vocatis rationibus, demonstratio ad finem perducitur, et quaesita series prodit

$$f(a+i) = fa + f'a.i + \frac{f''a}{1.2}.i^2 \ldots + \frac{f^n.a}{1\ldots n}.i^n \ldots$$

vel mutatis litteris

$$f(a+i) = fi + f'i.a + \frac{f''i}{1.2}a^2 \ldots + \frac{f^n i}{1.2\ldots n}a^n \ldots$$

§. 8.

Cum functionibus fa, $f(a+i)$ in series evolutis quasi fundamentum jactum sit, cui nunc omne Analyseos sublimioris aedificium superstruere liceat, ipsam hanc disquisitionem, qua istae evolutiones sunt peractae, fundamentalis disquisitionis nomine satis apte inscriptam a nobis esse apparet. Ob eandem autem harum formularum gravitatem, non abs re fore speravimus, aliam adhuc ejusdem rei expositionem publico mandare, quoniam propter faciliorem conclusionum nexum maxime ad hanc expositionem etiam tironibus aditus patet.

Notione differentialium (hucusque per $f'a$, $f''a$ . . . . designatorum, significantius autem per $\partial.fa$, $\partial^2.fa$ . . . . indicandorum), qualem supra statuimus, etiam hic retenta, statim hoc colligimus theorema:

„differentiale functionis $m.a^n$, n numerum designante integrum ac positivum, aequa-
„tur expressioni $n.m.a^{n-1}$,"

cujus demonstratio facile e theoremate binomiali pro exponente positivo et integro petenda est.

Hanc functionis $m.a^n$, dum in $n.m.a^{n-1}$ transit, alterationem, cum saepius in Analysi occurrat, cumque praeterea ad nostrum institutum expediat, hic in usum vocaturi sumus, ideoque ei, Krampio duce peculiare *derivationis* nomen tribuemus; unde statim sequitur theorema: „derivatio cujuspiam aggregati aequatur aggregato derivatio-„num singulorum terminorum."

§. 9.

Derivatio in eo igitur a differentiatione differt, quod cum illa non nisi ad formam ma^n applicari possit, haec contra ad omnem omnino functionem, quacunque forma gaudentem conferenda est.

Facillime autem perspicitur, eandem expressionem per hanc vel per illam mutandi rationem evasuram, quomodocunque functio proposita exhibeatur, sive forma gaudet, quae derivationis rationem admittat, sive secus se habet, ideoque soli differentiationi adaptanda est.

E nexu hujus propositionis cum theoremate in fine §^{hi} praecedentis allato ad hanc gravissimam conclusionem inducimur: „differentiale cujuspiam functionis forma-„liter designatae aequatur aggregato derivationum terminorum seriei algebraicae illi „functioni respondentis."

Exempli gratia, si ponitur:

$$Sin\,x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2..5} - \frac{x^7}{1.....7} \cdot \cdot \cdot \cdot$$

nobis persuasum habere possumus, differentiale ipsius $Sin\,x$ tale evasurum, quod in seriem explicatum, formam

$$1-\frac{x^2}{1.2}+\frac{x^4}{1..4}-\frac{x^6}{1....6}\ldots$$

induat, quippe quae e serie functioni $Sin\,x$ respondente per derivationem nascitur; parique modo evincitur, differentiale secundum functionis $Sin\,x$ fore

$$-x+\frac{x^3}{1.2.3}-\frac{x^5}{1...5}\ldots.$$

§. 10.

Haec ultima autem propositio omnem nobis offert apparatum ad utrumque problema solvendum.

Ponatur enim solito modo

$$fa=f\overset{0}{a}+\alpha a+\beta a^2+\gamma a^3\ldots.+n.\overset{n}{\alpha}.a^n\ldots.$$

sumptis dein differentialibus et derivationibus, illis a sinistra, his vero a dextra aequationis parte, tot nascuntur aequationes, quot unitates maximo exponenti ipsius a in serie ficta tribuuntur, vel quot sunt coefficientes determinandi; quarum aequationum singulae valorem coefficientis in fronte collocati, posito $a=0$, statim innuent.

Ponatur porro

$$f(a+i)=f\overset{0}{a}+\alpha(a+i)+\beta(a+i)^2+\gamma(a+i)^3\ldots.+\overset{n}{\alpha}(a+i)^n\ldots.$$

vel, multiplicationibus actu institutis,

$$\begin{aligned}f(a+i)=f\overset{0}{a}&+\alpha a+\beta a^2+\gamma a^3\ldots.+\overset{n}{\alpha}.a^n\ldots.\\&+i(\alpha+2\beta a+3\gamma a^2\ldots.+n\overset{n}{\alpha}.a^{n-1}\ldots.)\\&+i^2\left(\beta+3\gamma a\ldots.+\frac{n.n-1}{1.2}\overset{n}{\alpha}a^{n-2}\ldots.\right)\\&\quad\vdots\\&+i^n(\overset{n}{\alpha}\ldots.)\\&\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Cum autem suppositum sit $f(a+i)=f\overset{0}{a}+\alpha(a+i)\ldots.+\overset{n}{\alpha}(a+i)^n\ldots.$ erit ideo

$$fa=f\overset{0}{a}+\alpha a+\beta a^2+\gamma a^3\ldots.+\overset{n}{\alpha}a^n\ldots.$$

et dehinc, sumptis utrinque differentialibus et derivationibus primi, secundi etc. n^{ti} ordinis:

$$\begin{aligned}\partial.fa&=\alpha+2\beta a+3\gamma a^2\ldots.+n\overset{n}{\alpha}.a^{n-1}\ldots.\\\partial^2fa&=2\beta+2.3.\gamma.a\ldots.+n.n-1.\overset{n}{\alpha}a^{n-2}\ldots.\\\partial^3fa&=2.3.\gamma\ldots.+n.n-1.n-2\overset{n}{\alpha}a^{n-3}\ldots.\\&\vdots\\\partial^nfa&=n.n-1\ldots.[n-(n-1)]\overset{n}{\alpha}\ldots.=1.2.3\ldots.n.\overset{n}{\alpha}\ldots.\\&\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

e quibus expressionibus cum singulis factoribus, qui in formulis supra exhibitis consecutivas ipsius i potestates multiplicant, collatis, facile concludimus, et primum terminum functionis $f(a+i)$ evolutae esse fa, et coefficientem generalem in i^n ductum fore

$$\frac{\partial^n fa}{1.2.3\ldots.n}.$$

De evolvendis functionibus $f(a+\alpha i+\beta i^2\ldots.)$, $\varphi f a$ *).

§. 11.

Ad primum problema solvendum designetur summa terminorum $(\alpha i+\beta i^2\ldots.)$ per p; quo effecto, habemus secundum legem generalem §$^{\text{hi}}$ 7.:

$$f(a+p)=fa+p.\partial fa+p^2.\frac{\partial^2 fa}{1.2}\ldots.+p^n\frac{\partial^n fa}{1.2\ldots.n}\ldots.;$$

totum igitur negotium huc est reductum, ut e quantitatibus p, $p^2\ldots.p^n$ omnes colligantur termini, qui eandem ipsius i potestatem exhibent.

Si n numerum designat integrum ac positivum, satis constat esse:

$$(\alpha i+\beta i^2+\gamma i^3\ldots.)^k=\mathfrak{P}^k\overset{k}{C}.i^k+\mathfrak{P}^{k+1}\overset{k}{C}.i^{k+1}\ldots.+\mathfrak{P}^n\overset{k}{C}.i^n\ldots.,$$
$$(\overset{1}{\alpha},\overset{2}{\beta},\overset{3}{\gamma}\ldots.)$$

*) Functionum $f(a+\alpha i\ldots.)$, $\varphi f a$ evolutiones, quae etsi maximi sint momenti, in permultis tamen vel completioribus de calculo differentiali tractatibus plane omittuntur, tractatas invenimus in: *Kramp Arithmétique universelle, Cpt. XX.*, et in *Crelle, Sammlung mathematischer Aufsätze, zweiter Band, Berlin* 1822, *pag.* 175 sqq. Nos ad idem problema solvendum viam ingressi sumus, quam *La Croix* (*Traité etc. Pars I. pag.* 325) indicat; sicque, evolutione functionis $f(a+\alpha i\ldots.)$, e notissima combinatoria methodo petita in auxilium vocata, expressionem generalem et independentum coefficientis $\frac{\partial^n \varphi f a}{\partial a^n}$ e voto nacti sumus. Methodus a nobis hic usurpata in eo quidem reprehendi potest, quod generalius problema ad speciale revocat; cum autem in omni disciplina et praesertim in Mathesi, e nostra quidem sententia, generaliores casus specialibus subjicere et liceat, et conveniat, ubicunque ex hac institutione ad faciliorem disquisitionem nonnihil adjumenti evenit: speravimus fore ut methodus, quam hic secuti sumus, quippe qua disquisitio ipsa mirifice sublevatur non omnino generalioribus methodis postponatur.

ubi secundum notissimam signandi methodum C denotat combinationes, in quibus admissae sunt repetitiones ex elementis α, β, γ . . . .; huicque characteri exponens classis superscriptus est, exponens autem summae ad laevam superne adjunctus; littera $\mathfrak{P}$ monet, unamquamque complexionum per respondentem ei numerum permutationum multiplicandam esse.

Quodsi igitur in termino generali $\mathfrak{P}^n \overset{k}{C}.i^n$ pro k successive omnes valores 1, 2, 3 . . . . n substituuntur, quaesitam coefficientium quantitatis i^n polynomii $(\alpha i + \beta i^2 \ldots)$ paullatim ad potentiam 1^{mam}, 2^{dam} . . . . n^{tam} elevati summam reperimus esse $=$

$$\mathfrak{P}^n \overset{1}{C} + \mathfrak{P}^n \overset{2}{C} + \mathfrak{P}^n \overset{3}{C} \ldots . + \mathfrak{P}^n \overset{n}{C},$$

vel hanc concinnius expressam:

$$\Sigma \mathfrak{P}^n \overset{k}{C} (k = 1,\ 2,\ 3 \ldots . n),$$
$$(\overset{1}{\alpha},\ \overset{2}{\beta},\ \overset{3}{\gamma} \ldots .)$$

ubi littera Σ nos monet, aggregatum omnium terminorum, qui paullatim e substitutione 1, 2 . . . . n pro k in expressione generali prodeunt, sumendum esse.

In serie autem $fa + p.\partial fa \ldots + \frac{p^n \partial^n fa}{1.2 \ldots . n} \ldots .$ omnes termini ipsius p, cui $k = 1$ respondet, in ∂fa, et generaliter termini ipsius p^n valori $k = n$ respondentis in $\frac{\partial^n fa}{1.2 \ldots . n}$ sunt ducti; his ergo factoribus restitutis, quantitatis i^n coefficiens generalis erit

$$\Sigma \frac{\mathfrak{P}^n \overset{n}{C}.\partial^k fa}{1.2.3 \ldots . k} (k = 1,\ 2 \ldots . n)$$
$$(\overset{1}{\alpha},\ \overset{2}{\beta},\ \overset{3}{\gamma} \ldots .).$$

Haec autem expressio plane problema nostrum solvit, quia nos ad cujusvis termini functionis $f(a + \alpha i \ldots .)$ evolutae, in datam n^{tam} potentiam quantitatis i ducti computum manuducit; hoc enim tantummodo opus erit, ut in inventa expressione pro n assumatur ille valor numericus, excepto quidem valore $n = 0$ primum seriei terminum innuente, quem autem esse $= fa$ jam aliunde constat; hunc enim valorem colligimus, ponendo in $f(a + \alpha i \ldots .)\, i = 0$.

§. 12.

His peractis ad generalius problema solvendum aggredimur, quo scilicet postulatur, ut functio maxime universalis φfa in seriem secundum potestates integras ipsius a progredientem evolvatur.

Cum in hoc casu φfa functio sit ipsius fa, ipsa autem fa ad primariam suam quantitatem a referatur, ideoque φfa definitam quoque habeat dependentiam a quantitate a: tria discernenda sunt differentialium genera

primum: $\frac{\partial^n \varphi f a}{(\partial f a)^n}$ differentiale n^{tum} functionis $\varphi f a$ respectu $f a$;

secundum: $\frac{\partial^n \varphi f a}{(\partial a)^n}$ differentiale ejusdem functionis respectu a;

tertium: $\frac{\partial^n f a}{(\partial a)^n}$ differentiale quantitatis $f a$ proxime primariae, respectu remotioris a.

Formula $\S^{\text{hi}}$ 6. statim nobis offert:

$$\varphi f a = \varphi f \overset{\circ}{a} + a . \frac{\partial \varphi f \overset{\circ}{a}}{\partial a} + \frac{a^2}{1.2} . \frac{\partial^2 \varphi f \overset{\circ}{a}}{\partial a^2} \ldots . + \frac{a^n}{1 \ldots . n} . \frac{\partial^n \varphi f \overset{\circ}{a}}{(\partial a)^n} \ldots . ;$$

quae autem ex hujus seriei notitia nobis redundaret utilitas, ea parvi esset momenti, nisi disquisitionem ulterius persequeremur; supposuimus enim hic, dari quidem ipsius $\varphi f a$ a $f a$ et pari modo ipsius $f a$ a quantitate a dependentiam, unde etiam functiones

$$\frac{\partial \varphi f a}{\partial f a}, \ \frac{\partial^2 \varphi f a}{(\partial f a)^2} \ldots . ; \ \frac{\partial f a}{\partial a}, \ \frac{\partial^2 f a}{(\partial a)^2} \ldots .$$

innotescunt; relationem autem inter $\varphi f a$ et ipsam quantitatem a adhuc ignoramus, ideoque etiam differentialia hujus formae $\frac{\partial^n \varphi f a}{(\partial a)^n}$ incognita censenda sunt, et formula tradita coefficientes omnino incognitos continet. Gravior ergo pars solutionis adhuc restat, illa nempe, in qua disquirendum est, quomodo expressio generalis $\frac{\partial^n \varphi f a}{(\partial a)^n}$ seriei allatae a functionibus cognitis $\frac{\partial^k \varphi f a}{(\partial f a)^k}$, $\frac{\partial^k f a}{(\partial a)^k}$ pendeat, ita ut illam per has exprimere valeamus; quae autem investigatio ex iis, quae jam in praecedentibus peractae sunt evolutionibus facillime nobis petenda erit.

Ponatur enim quantitatem a abire in $(a + i)$, ac prodibit

$$\varphi f (a + i) = \varphi \left\{ f a + i . \frac{\partial f a}{\partial a} + \frac{i^2}{1.2} . \frac{\partial^2 f a}{(\partial a)^2} \ldots . \right\} \quad (\mathcal{A})$$

cujus seriei terminum generalem jam constat esse

$$\Sigma \frac{\mathfrak{P}^n \overset{k}{C} . \frac{\partial^k \varphi f a}{(\partial f a)^k}}{1.2.3 \ldots . k} i^n, \quad (k = 1 \ldots . n)$$

$$\left(\overset{1.}{\frac{\partial f a}{\partial a}}, \ \overset{2.}{\frac{\partial^2 f a}{1.2.(\partial a^2)}}, \ \overset{3.}{\frac{\partial^3 f a}{1.2.3.(\partial a)^3}} \ldots . \right).$$

Cum autem supra jam expositum sit, expressionem $\varphi f a$ functionem esse ipsius a, definitam hanc dependentiam peculiari signo hic exprimere licebit, ita ut sit: $\varphi f a = \psi a$, ideoque $\varphi f (a + i) = \psi (a + i)$; unde subsidio evolutionis generalis $\S^{\text{hi}}$ 7. exoritur aequatio:

$$\psi(a+i) = \psi a + i \cdot \frac{\partial \psi a}{\partial a} + \frac{i^2}{1.2} \cdot \frac{\partial^2 \psi a}{(\partial a)^2} \cdots + \frac{i^n}{1.2 \ldots . n} \cdot \frac{\partial^n \psi a}{(\partial a)^n} \ldots . (B);$$

Habemus igitur in (A) et (B) duas ejusdem quantitatis $\varphi f(a+i) = \psi(a+i)$ expressiones; jam si coefficientes ejusdem potestatis quantitatis i, secundum quam utraque series ordinatur, inter se aequales statuuntur, haec resultat formula:

$$\frac{\partial^n \varphi f a}{1.2 \ldots . n (\partial a)^n} = \Sigma \frac{\mathfrak{P}^n \overset{k}{C} \dfrac{\partial^k \varphi f a}{(\partial f a)^k}}{1.2 \ldots . k}, \ (k = 1,\ 2 \ldots . n)$$

$$\left(\overset{1}{\frac{\partial f a}{\partial a}},\ \overset{2}{\frac{\partial^2 f a}{1.2 (\partial a)^2}},\ \overset{3}{\frac{\partial^3 f a}{1.2.3 (\partial a)^3}}, \ldots .\right)$$

quae erat quaesita.

§. 13.

Exemplum.

Sit $n = 6$, ideoque $\dfrac{\partial^6 \varphi f a}{(\partial a)^6}$ functio evolvenda.

Ponatur jam brevitatis gratia $f' a$ pro $\dfrac{\partial f a}{\partial a}$, et generaliter $f^n a$ pro $\dfrac{\partial^n f a}{(\partial a)^n}$; similique modo $\varphi^n f a$ locum teneat expressionis $\dfrac{\partial^n \varphi f a}{(\partial f a)^n}$, quibus institutis, expressio

$$\Sigma \frac{1.2.3.4.5.6}{1.2 \ldots . k} \mathfrak{P}^6 \overset{k}{C} \varphi^k f a$$

$$\overset{1}{f' a},\ \overset{2}{\frac{f'' a}{1.2}},\ \overset{3}{\frac{f''' a}{1.2.3}},\ \overset{4}{\frac{f^{\text{IV}} a}{1 \ldots 4}},\ \overset{5}{\frac{f^{\text{V}} a}{1 \ldots 5}},\ \overset{6}{\frac{f^{\text{VI}} a}{1.2 \ldots 6}}$$

supputanda erit, pro k successive numeris ab 1 usque ad 6 substitutis.

At primo quidem complexiones per $\Sigma \mathfrak{P}^6 \overset{k}{C}$ exhibitae et ad elementa 1, 2, 3, 4, 5, 6 spectantes actu evolutae erunt:

$$\mathfrak{P}^6 \overset{1}{C} = 6$$
$$\mathfrak{P}^6 \overset{2}{C} = 2(1.5) + 2(2.4) + (3.3)$$
$$\mathfrak{P}^6 \overset{3}{C} = 3(1.1.4) + 6(1.2.3) + (2.2.2)$$
$$\mathfrak{P}^6 \overset{4}{C} = 4(1.1.1.3) + 6(1.1.2.2)$$
$$\mathfrak{P}^6 \overset{5}{C} = 5(1.1.1.1.2)$$
$$\mathfrak{P}^6 \overset{6}{C} = (1.1.1.1.1.1);$$

proinde habemus:

pro $k = 1$

$$f^{\text{VI}} a \cdot \varphi' f a,$$

pro $k=2$

$$\frac{1\ldots.6}{1.2}\left\{2f'a.\frac{f^{\text{v}}a}{1\ldots5}+\frac{2f''a}{2}.\frac{f^{\text{IV}}a}{1..4}+\left(\frac{f'''a}{1.2.3}\right)^2\right\}\varphi''fa,$$

$$=\left\{6f'af^{\text{v}}a+15f''af^{\text{IV}}a+10f'''af'''a\right\}\varphi''fa;$$

pro $k=3$

$$\frac{1\ldots.6}{1.2.3}\left\{3(f'a)^2\frac{f^{\text{IV}}a}{1.2.3.4}+6f'a\frac{f''a}{1.2}.\frac{f'''a}{1.2.3}+\left(\frac{f''a}{2}\right)^3\right\}\varphi'''fa$$

$$=\left\{15(f'a)^2f^{\text{IV}}a+60f'a.f''a.f'''a+15(f''a)^3\right\}\varphi'''fa;$$

pro $k=4$

$$\frac{1\ldots.6}{1.2.3.4}\left\{4(f'a)^3.\frac{f'''a}{1.2.3}+6(f'a)^2.\frac{(f''a)^2}{4}\right\}\varphi^{\text{IV}}fa$$

$$=\left\{20(f'a)^3f'''a+45(f'a)^2(f''a)^2\right\}\varphi^{\text{IV}}fa;$$

pro $k=5$

$$\frac{1\ldots.6}{1.2.3.4.5}\left\{5(f'a)^4\frac{f''a}{2}\right\}\varphi^{\text{v}}fa=15(f'a)^4f''a\,\varphi^{\text{v}}fa;$$

pro $k=6$

$$(f'a)^6\varphi^{\text{VI}}fa.$$

Summa igitur harum expressionum, quaesitum ipsius $\frac{\partial^6\varphi fa}{(\partial a)^6}$ exhibet valorem.

§. 14.

Quem hic secundum formulam generalem exhibuimus computum, eum directa etiam methodo instituere possumus, si in auxilium vocamus propositiones notas has:

1) $\frac{\partial\varphi fa}{\partial a}=\partial.\frac{\varphi fa}{\partial fa}.\frac{\partial fa}{\partial a}.$

2) $\partial\{\psi a.\psi'a.\psi''a\ldots.\}=\partial.\psi a.\psi'a.\psi''a\ldots.+\partial.\psi'a\psi a\psi''a\ldots.+\ldots.$

Ponatur enim $\frac{\partial.\varphi fa}{\partial fa}=z'$, $\frac{\partial fa}{\partial a}=y'$; designentur porro harum functionum differentialia per z'', z''' etc., y'', y''' etc. illa respectu fa sumpta, haec vero ad quantitatem a primariam relata; quibus institutis, erit $\frac{\partial\varphi fa}{\partial a}=z'y'$, quae expressio iterum differentiata talis prodibit $z''y'2+z'y''$; quod si paullatim ad altiora differentialia ascendimus, has post legitimam membrorum reductionem nanciscimur aequalitates:

$$\frac{\partial\varphi fa}{\partial a}=z'y',$$

$$\frac{\partial^2\varphi fa}{(\partial a)^2}=z''y'^2+z'y'',$$

$$\frac{\partial^3\varphi fa}{(\partial a)^3}=z'''y'^3+z''.3y'y''+z'y''',$$

$$\frac{\partial^4 \varphi f a}{(\partial a)^4} = z^{\mathrm{IV}} y'^4 + z'''.\,6 y'^2 y'' + z''(4 y' y''' + 3 y''^2) + z' y^{\mathrm{IV}},$$

$$\frac{\partial^5 \varphi f a}{(\partial a)^5} = z^{\mathrm{V}}.\,y'^5 + z^{\mathrm{IV}}.\,10 y'^3 y'' + z'''(10 y'^2 y''' + 15 y' y''^2) + z''(5 y' y^{\mathrm{IV}} + 10 y'' y''') + z' y^{\mathrm{V}},$$

$$\frac{\partial^6 \varphi f a}{(\partial a)^6} = z^{\mathrm{VI}} y'^6 + z^{\mathrm{V}}.\,15 y'^4 y'' + z^{\mathrm{IV}}(20 y'^3 y''' + 45 y'^2 y''^2) + z'''(15 y'^2 y^{\mathrm{IV}} + 60 y' y'' y''' + 15 y''^3) + z''(6 y' y^{\mathrm{V}} + 15 y'' y^{\mathrm{IV}} + 10 y'''^2) + z' y^{\mathrm{VI}}$$

quarum ultima cum illa in §$^{\text{ho}}$ praecedenti inventa plane congruit.

De aliquot formulis e generali expressione $\Sigma \frac{\mathfrak{P}^n \overset{k}{C}\, \partial^k f a}{1.2\ldots.k}$ petendis.

---

§. 15.

Formularum generalium gravis admodum utilitas in eo posita est, quod hae formulae ad omnes omnino quaestiones, ubi de definita quomodocunque functione agitur in seriem evolvenda, applicari possunt; ex uberrima igitur harum applicationum copia, quatenus per fines juste ponendos licebat, aliquot, quae quidem ad formulam allatam spectent, hoc loco tractare haud alienum judicavimus.

De evolvenda functione $(a x + b x^2 + c x^3 \ldots)^m$.

§. 16.

Sit primum terminus generalis polynomii $(a + b x \ldots.)$ ad potestatem m^{tam} elevati quaerendus,

Hic habemus

$$f a = a^m.$$

$$\partial f a = m\,.\,a^{m-1} = {}^m\overset{1}{\mathfrak{B}}\,.\,m^{m-1},$$

$$\partial^2 f a = m\,.\,m-1\,.\,a^{m-2} = 1.2\,.\,{}^m\overset{2}{\mathfrak{B}}\, a^{m-2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\partial^k f a = m\,.\,m-1 \ldots.(m-k+1) a^{m-k} = 1.2\ldots.k\, {}^m\overset{k}{\mathfrak{B}} a^{m-k},$$

quibus valoribus in generali expressione substitutis, quaesitus terminus erit:

$$\Sigma^m \overset{k}{\mathfrak{B}}.\,a^{m-k}\,\mathfrak{P}^n\overset{k}{C}.\,x^n,\quad (k=1,\ 2,\ \ldots .\ n)$$
$$(\overset{1}{b},\ \overset{2}{c},\ \overset{3}{d}\ \ldots .).$$

Jam si series proposita formam habet $(ax+bx^2\ldots .)^m$, statim raducitur ad hanc $x^m(a+bx\ldots .)^m$, ideoque solutio eadem manet, hac tantum restrictione, ut expressio inventa nunc in x^{m+n} ducenda sit.

§. 17.

Si m numerum designat integrum ac positivum, coefficientem generalem functionis $(a+bx+cx^2\ldots .)^m$ satis constat esse:

$$\mathfrak{P}\,{}^{m+n}\overset{m}{C}$$
$$(\overset{1}{a},\ \overset{2}{b},\ \overset{3}{c}\ \ldots .),$$

quae expressio cum in expressione maxime generali §^hi praecedentis, quippe quae ad omnes omnino valores, quos quidem m induere possit, adtinet, subsit oporteat, hanc consequimur aequalitatem

$$\underset{(\overset{1}{a},\ \overset{2}{b},\ \overset{3}{c}\ldots .)}{\mathfrak{P}\,{}^{m+n}\overset{m}{C}} = \underset{(\overset{1}{b},\ \overset{2}{c},\ \overset{3}{d}\ldots .)}{\Sigma^m\overset{k}{\mathfrak{B}}\,a^{m-k}\mathfrak{P}^n\overset{k}{C}}$$

$(k=1,\ 2\ \ldots .\ n)$, vel $(k=1,\ 2\ \ldots .\ m)$ pro $m<n$.

Ad hanc aequalitatem demonstrandam talem ingressi sumus viam *).

E deductione ipsa formarum combinatoriarum liquet, formam

$${}^{m+n}\overset{m}{C}(\overset{1}{a},\ \overset{2}{b},\ \overset{3}{c}\ \ldots .)$$

constare e formis:

$$a^{m-1}\,{}^n\overset{1}{C} + a^{m-2}\,{}^n\overset{2}{C}\ \ldots .\ + a^{m-k}\,{}^n\overset{k}{C}\ \ldots .$$

ad elementa $(\overset{1}{b},\ \overset{2}{c},\ \overset{3}{d}\ \ldots .)$ relatis; unde etiam

$$\underset{(\overset{1}{a},\ \overset{2}{b},\ \overset{3}{c}\ldots .)}{\mathfrak{P}\,{}^{m+n}\overset{m}{C}} = \underset{(\overset{1}{b},\ \overset{2}{c},\ \overset{3}{d}\ldots .)}{\Sigma\mathfrak{P}\,a^{m-k}\,{}^n\overset{k}{C}} \begin{Bmatrix} k=1,\ 2\ \ldots .\ n \\ k=1,\ 2\ \ldots .\ m \end{Bmatrix}.$$

Complexionum ad dextram aequationis partem designatarum generalis forma est $a^{m-k}.\,b^q.\,c^r.\,d^s\ldots .$, ita ut sit: $q+r+s\ldots .=k$, $q+2r+3s\ldots .=n$, cujus igitur formae permutationum numerus exhibetur per

---

*) Multae egregiae consecutiones ex hac identitate fluentes inveniuntur in *Weingärtner Lehrbuch der combinatorischen Analysis, zweiter Theil p.* 174 *sqq.* Nostrum autem est, hanc formarum congruentiam ex ipsa, qua compositae sunt ratione, hic demonstrare.

$$\frac{(m-k+q+r+s\ldots.)!}{(m-k)!\,q!\,r!\,s!\ldots.} = \frac{m!}{(m-k)!\,q!\,r!\,s!\ldots.}$$

(signo $n!$ pro $1.2.3\ldots.n$ adhibito, quod notum est).

Haec ultima expressio, factoribus numeratori et denominatori communibus deletis, statim reducitur in

$$\frac{m(m-1)\ldots.(m-k+1)}{q!\,r!\,s!\ldots.} = \frac{{}^m\overset{k}{\mathfrak{B}}.k!}{q!\,r!\,s!\ldots.}.$$

Fractio autem $\frac{k!}{q!\,r!\,s!\ldots.}$ est permutationum numerus complexionis $b^q.c^r.d^s\ldots.$, sive formae ${}^n\overset{k}{C}$; adest igitur aequatio

$$\underset{(\overset{1}{b},\ \overset{2}{c},\ \overset{3}{d}\ \ldots)}{\mathfrak{P}\, a^{m-k}\,{}^n\overset{k}{C} = {}^m\overset{k}{\mathfrak{B}}.a^{m-k}\mathfrak{P}\,{}^n\overset{k}{C},}$$

ideoque etiam

$$\Sigma\mathfrak{P}\, a^{m-k}\,{}^n\overset{k}{C} = \Sigma\,{}^m\overset{k}{\mathfrak{B}}\, a^{m-k}\,\mathfrak{P}\,{}^n\overset{k}{C}$$

$$(k=1,\ 2\ \ldots.\ n),\ \text{vel}\ (k=1,\ 2\ \ldots.\ m)$$

et denique secundum relationem supra jam probatam

$$\underset{(\overset{1}{a},\ \overset{2}{b},\ \overset{3}{c}\ldots.)}{\mathfrak{P}^{m+n}\overset{m}{C}} = \underset{(\overset{1}{b},\ \overset{2}{c},\ \overset{3}{d}\ldots.)}{\Sigma\,{}^m\overset{k}{\mathfrak{B}}\, a^{m-k}\,\mathfrak{P}\,{}^n\overset{k}{C}} \begin{Bmatrix} k=1,\ 2\ \ldots.\ n \\ k=1,\ 2\ \ldots.\ m \end{Bmatrix}.$$

De evolvendis functionibus: $\overset{e}{log}(1+ax\ldots.)$, $e^{ax+bx^2\ldots.}$.

§. 18.

Sit primum functio $\overset{e}{log}(m+ax+bx^2\ldots.)$ evolvenda.

Hic habemus

$$fm = \log m$$

$$\partial.\log m = \frac{1}{m}$$

$$\partial^2 \log m = -\frac{1}{m^2}$$

$$\partial^3 \log m = +\frac{1.2}{m^3},$$

$$\ldots\ldots\ldots$$

$$\partial^k \log m = \frac{1.2.3\ldots.(k-1)(-1)^{k-1}}{m^k} = \frac{1.2\ldots.k(-1)^{k-1}}{k.m^k},$$

hinc seriei terminus generalis erit:

$$\Sigma \frac{\mathfrak{P}^n \overset{k}{C} (-1)^{k-1} . 1 . 2 \dots . k}{k . 1 . 2 \dots . k . m^k} x^n, \ (k = 1, \dots . n)$$

$$(\overset{1}{a}, \overset{2}{b}, \overset{3}{c} \dots .)$$

ponendo autem $m = 1$, quo functio ipsa in $\overset{e}{log}(1 + a x \dots .)$ mutatur, expressio inventa transit in

$$\Sigma \frac{\mathfrak{P}^n \overset{k}{C} (-1)^{k-1}}{k} x^n, \ (k = 1. \ 2 \dots . n)$$

$$(\overset{1}{a}, \overset{2}{b}, \overset{3}{c} \dots .).$$

Quodsi nunc functio $log(1 + \alpha x)$ proposita est, statim elucet, nullum alium valorem pro k assumi posse nisi $k = n$, quam ob rem terminus generalis hoc in casu exhibetur per

$$\pm \frac{1}{n} a^n x^n,$$

ita ut series actu evoluta hoc modo se habitura sit:

$$\overset{e}{log}(1 + a x) = a x - \frac{a^2 x^2}{2} + \frac{a^3 x^3}{3} \dots . \pm \frac{a^n x^n}{n} \dots .$$

§. 19.

Proposita nunc sit functio $e^{m + ax + bx^2} \dots .$ in seriem evolvenda; e comparatione expressionis nostrae cum $f(a + \alpha i \dots .)$ prodit

$$fa = e^m, \ i = x, \ \partial^k f a = e^m,$$

quibus substitutis, adest terminus generalis

$$\Sigma \frac{\mathfrak{P}^n \overset{k}{C} . e^m}{1 \dots . k} x^n,$$

quae expressio pro $m = o$, quo functio evolvenda formam $e^{ax + bx^2} \dots .$ induit mutatur in

$$\Sigma \frac{\mathfrak{P}^n \overset{k}{C}}{1 \dots . k} x^n, \ (k = 1, \ 2, \dots . n)$$

$$(\overset{1}{a}, \overset{2}{b}, \overset{3}{c} \dots .).$$

De functionibus: $log\, n . Sin\, x$, $log\, n . Cos\, x$ evolvendis.

§. 20.

Cum sit $Sin\, x = x - \frac{x^3}{1.2.3} + \dots .$ erit

$$log\, Sin\, x = log\, x + log\left(1 - \frac{x^2}{1.2.3} + \frac{x^4}{1 \dots 5} - \dots .\right)$$

ponendo igitur in ultima expressione $x^2 = z$, terminus ejus generalis secundum formulam §hi 18. erit:

$$\Sigma \frac{\mathfrak{P}^{\frac{n}{2}} \overset{k}{C} (-1)^{k-1}}{k} x^n, \quad \left(k = 1, 2, 3 \ldots . \frac{n}{2}\right)$$

$$\left(-\frac{\overset{1}{1}}{1.2.3} + \frac{\overset{2}{1}}{1\ldots 5} - \frac{\overset{3}{1}}{1\ldots . . 7} \ldots\right)$$

n numero existente pari *).

Summa ergo coefficientium ipsius x^n constat e terminis

$$\left.\begin{array}{l} + \mathfrak{P}^{\frac{n}{2}} \overset{1}{C} \\ - \frac{1}{2} \mathfrak{P}^{\frac{n}{2}} \overset{2}{C} \\ + \frac{1}{3} \mathfrak{P}^{\frac{n}{2}} \overset{3}{C} \\ \vdots \\ \pm \frac{1}{n} \mathfrak{P}^{\frac{n}{2}} \overset{n}{C} \end{array}\right\} \left(-\frac{\overset{1}{1}}{1.2.3} + \frac{\overset{2}{1}}{1\ldots 5} \ldots\right)$$

de quibus autem haec notanda sunt:

Pro $\frac{n}{2}$ numero pari omnes complexiones hic exhibitae valorem habebunt absolute positivum; sin autem $\frac{n}{2}$ impar fuerit, omnes complexiones per se negativae evadent. In utroque igitur casu signa terminorum alternabuntur, ita quidem, ut primus terminus positivus assumendus sit pro $\frac{n}{2}$ numero pari, negativus autem pro impari.

---

*) E formula §hi 18. continuo ad nostrum casum applicata terminus generalis prodit

$$\Sigma \frac{\mathfrak{P}^{n} \overset{k}{C} (-1)^{k-1}}{k} x^n, \quad (k = 1, 2 \ldots . n)$$

$$\left(\overset{1}{0}, -\frac{\overset{2}{1}}{1.2.3}, \overset{3}{0}, +\frac{\overset{4}{1}}{1\ldots 5}, \overset{5}{0}, -\frac{\overset{6}{1}}{1\ldots . . 7} \ldots .\right);$$

cum autem hic omnes complexiones, quarum exponens summae numerus est impar, per se sunt $= 0$, consequimur, nullum pro n valorem imparem assumi posse. Insuper vero aeque facile nobis persuadetur, omnes complexiones hic evolvendas, quarum exponens classis major sit quam $\frac{n}{2}$ itidem evanescere; qua de causa igitur plane supervacaneum erit, pro k valorem assumere majorem quam $\frac{n}{2}$. Quodsi nunc elementa imparibus indicibus praedita plane omittuntur, nobis licebit, tam exponentis n quam indicum 2, 4, 6 . . . . dimidium sumere, quo tandem effecto in priorem relabimur formulam.

Quod ut ex expressione ipsa evadat, nullo respectu habito signorum quantitatum, e quibus complexiones formantur, hoc est, his quantitatibus omnino positivis sumendis, hanc pro termino generali instituimus formulam:

$$\Sigma \frac{\mathfrak{P}^{\frac{n}{2}}\overset{k}{C}(-1)^{\frac{n}{2}+k-1}}{k}\, x^n, \quad \left(k = 1,\ 2,\ \ldots\ \frac{n}{2}\right)$$

$$\left(\overset{1}{\frac{1}{1.2.3}},\ \overset{2}{\frac{1}{1\ldots5}},\ \overset{3}{\frac{1}{1\ldots\ldots7}},\ \overset{4}{\frac{1}{1\ldots\ldots\ldots9}}\ \ldots\right).$$

Ex. gr. coefficiens ipsius x^8 erit

$$\mathfrak{P}^4\overset{1}{C} - \tfrac{1}{2}\mathfrak{P}^4\overset{2}{C} + \tfrac{1}{3}\mathfrak{P}^4\overset{3}{C} - \tfrac{1}{4}\mathfrak{P}^4\overset{4}{C} =$$

$$\frac{1}{1\ldots\ldots\ldots9} - \tfrac{1}{2}\left(\frac{1}{1\ldots5}\right)^2 - \tfrac{1}{2}.2.\frac{1}{1.2.3}.\frac{1}{1\ldots\ldots7} + \tfrac{1}{3}.3\left(\frac{1}{1.2.3}\right)^2.\frac{1}{1\ldots5} - \tfrac{1}{4}\left(\frac{1}{1.2.3}\right)^4.$$

vel si omnes termini ad communem denominatorem rediguntur

$$\frac{5 - 63 - 60 + 420 - 350}{5(1.2.3.4.5.6.7.8.9)} = \frac{-48}{1\ldots\ldots\ldots9} = -\frac{1}{2^3\, 3^3\, 5^2\, 7}.$$

Cum sit $\log n\ Cos\, x = \log n\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1..4}\ \ldots\right)$ terminus generalis hujusce functionis in seriem evolutae, iisdem ut supra adhibitis rationibus, erit:

$$\Sigma \frac{\mathfrak{P}^{\frac{n}{2}}\overset{k}{C}(-1)^{\frac{n}{2}+k-1}}{k}\, x^n, \quad \left(k = 1,\ 2,\ \ldots\ \frac{n}{2}\right)$$

$$\left(\overset{1}{\frac{1}{1.2}},\ \overset{2}{\frac{1}{1..4}},\ \overset{3}{\frac{1}{1\ldots.6}}\ \ldots\right).$$

De summa potestatum per functiones combinatorias radicum exhibenda.

§. 21.

Si productum $(1 + ax)(1 + bx)(1 + cx)\ldots$ seriei $1 + \alpha x + \beta x^2 \ldots$ aequamus, coefficientes $\alpha, \beta, \gamma\ \ldots$ secundum legem notissimam hoc modo definiuntur, ut sit

$$\left.\begin{array}{l} \alpha = [\overset{1}{C}] \\ \beta = [\overset{2}{C}] \\ \gamma = [\overset{3}{C}] \\ \quad\vdots \\ \overset{n}{\alpha} = [\overset{n}{C}] \end{array}\right\} (a,\ b,\ c\ \ldots)$$

ubi per litteram $[C]$ uncinis inclusam combinationes omissis repetitionibus designamus.

E praecedentibus autem habemus

$$log(1+ax) = ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3} \dots$$

$$log(1+bx) = bx - \frac{b^2x^2}{2} + \frac{b^3x^3}{3} \dots$$

$$\dots\dots\dots\dots\dots\dots$$

quae aequationes, si summantur, progignunt hanc:

$$log(1+ax) + log(1+bx) \dots = (a+b\dots)x - \frac{(a^2+b^2\dots)}{2}x^2 \dots$$

vel

$$log(1+\alpha x+\beta x^2\dots) = S_1 x - \tfrac{1}{2}S_2 x^2 \dots \mp \frac{1}{n}S_n x^n \dots$$

(signo S_n pro $a^n + b^n + c^n \dots$ adhibito)

quod si denique coefficientes in eandem ipsius x potestatem ducti seorsim aequantur, habetur generaliter:

$$\mp \frac{1}{n}S_n = \Sigma\, \frac{\mathfrak{P}^n\overset{k}{C}(-1)^{k-1}}{k},\ (k=1,\ 2,\ \dots\ n)$$

$$(\overset{1}{\alpha},\ \overset{2}{\beta},\ \overset{3}{\gamma}\dots)\ \text{sive}\ \left\{\begin{matrix}[\overset{1}{C^1}],\ [\overset{2}{C^2}],\ [\overset{3}{C^3}]\dots \\ (a,\ b,\ c\dots)\end{matrix}\right\}$$

superiori nempe signo valente pro n numero pari, inferiori autem pro impari.

Poterimus autem formulam inventam ita comparare, ut illa ambiguitas plane removeatur; si enim pro k omnes valores, non ut re vera fecimus de 1 usque ad n, sed inverso ordine de n usque ad 1 ponimus, prodit

$$\frac{1}{n}S_n = \Sigma\frac{\mathfrak{P}^n\overset{k}{C}}{k},\ (k=n,\ n-1,\ n-2,\ \dots 1);$$

qua quidem in formula tenendum est, terminos secundae partis aequationis, ut paullatim eruuntur, alternantibus signis sumendos esse, a signo $(+)$ incipiendo.

Quod cum idem eventurum sit, si elementa ipsa alternantibus signis statuuntur, haec denique nobis se offert formula:

$$\frac{1}{n}S_n = \Sigma\,\frac{\mathfrak{P}^n\overset{k}{C}}{k}\left\{\begin{matrix}k=n,\ n-1\dots 1 \\ \text{sive} \\ k=1,\ 2\dots n\end{matrix}\right\}$$

$$\left\{\begin{matrix}[\overset{1}{C^1}],\ -[\overset{2}{C^2}],\ +[\overset{3}{C^3}]\dots \\ (a,\ b,\ c\dots)\end{matrix}\right\}.$$

Sit exempli gratia summa biquadratorum $a^4+b^4+c^4$ e functionibus $\alpha=a+b+c$, $\beta=ab+ac+bc$, $\gamma=abc$ struenda; habemus:

$$\left.\begin{array}{r} \frac{1}{4}\,\mathfrak{P}\,{}^4\overset{4}{C} \\ \frac{1}{3}\,\mathfrak{P}\,{}^4\overset{3}{C} \\ \frac{1}{2}\,\mathfrak{P}\,{}^4\overset{2}{C} \\ \mathfrak{P}\,{}^4\overset{1}{C} \end{array}\right\} (\overset{1}{\alpha}, -\overset{2}{\beta}, \overset{3}{\gamma}) \begin{array}{l} = \frac{1}{4}\alpha^4 \\ = -\alpha^2\beta \\ = \alpha\gamma + {}^{1}\beta^2 \\ = -\delta = 0 \end{array}$$

ideoque

$$S_4 = (a+b+c)^4 - 4(a+b+c)^2(ab+ac+bc) + 4(a+b+c)abc + 2(ab+ac+bc)^2.$$

§. 22.

Adsint nunc duo elementa $(\overset{1}{\alpha}, \overset{2}{\beta})$, e quibus omnes combinationes ad summam n evolvendae sint, eritque

$$\underset{(\overset{1}{\alpha},\ \overset{2}{\beta})}{\Sigma\mathfrak{P}\,{}^n\overset{k}{C}} = \alpha^n + (n-1)\alpha^{n-2}\beta + \frac{(n-2)}{1}\frac{(n-3)}{2}\alpha^{n-4}\beta^2 + \frac{(n-3)}{1}\frac{(n-4)}{2}\frac{(n-5)}{3}\alpha^{n-6}\beta^3 \ldots \mp \beta^{\frac{n}{2}}$$

pro n numero pari, pro impari autem erit

$$\underset{(\overset{1}{\alpha},\ \overset{2}{\beta})}{\Sigma\mathfrak{P}\,{}^n\overset{k}{C}} = \alpha^n + (n-1)\alpha^{n-2}\beta + \frac{(n-2)}{1}\frac{(n-3)}{2}\alpha^{n-4}\beta^2 + \frac{(n-3)}{1}\frac{(n-4)}{2}\frac{(n-5)}{3}\alpha_n^{-6}\beta^3 \ldots.$$

$$\ldots \pm \frac{1.2\ldots.\frac{n+1}{2}}{1.2\ldots\frac{n-1}{2}}\,\alpha\,\beta^{\frac{n-1}{2}},$$

quae ultima expressio aequatur $\frac{n+1}{2}\alpha\,\beta^{\frac{n-1}{2}}$.

Quodsi igitur formulam

$$\frac{1}{n}S_n = \underset{(\overset{1}{\alpha},\ -\overset{2}{\beta})}{\Sigma\frac{\mathfrak{P}\,{}^n\overset{k}{C}}{k}}, \quad (k = n,\ n-1,\ \ldots.\ 1)$$

ad hunc casum applicamus, prodit:

$$a^n + b^n = \alpha^n - n\alpha^{n-2}\beta + \frac{n.n-3}{1.2}\alpha^{n-4}\beta^2 - \frac{n.n-4.n-5}{1.2.3}\alpha^{n-6}\beta^3 \ldots. \pm 2\beta^{\frac{n}{2}}$$

et

$$a^n + b^n = \alpha^n - n\alpha^{n-2}\beta + \frac{n.n-3}{1.2}\alpha^{n-4}\beta^2 - \frac{n.n-4.n-5}{1.2.3}\alpha^{n-6}\beta^3 \ldots. \pm n\,\alpha\,\beta^{\frac{n-1}{2}}$$

prout n numerus sit par vel impar *).

*) Fusius haec res tractata invenitur in *Kramp Arithmétique universelle* Cpt. XXI.

§. 23.

Resumatur nunc aequatio

$$\log(1+\alpha x+\beta x^2\ldots.)=S_1 x-\tfrac{1}{2}S_2 x^2+\tfrac{1}{3}S_3 x^3\ \ldots.$$

Jam apparet, etiam eos numeros aequales futuros, qui istas expressiones pro logarithmis, basi eadem quidem sed ad lubitum sumpta, habent; hoc est:

$$(1+\alpha x+\beta x^2\ldots.)=e^{S_1 x-\frac{1}{2}S_2 x^2,\ldots}$$

quodsi igitur utrinque coefficientes ejusdem ipsius x potestatis inter se aequales statuuntur, prodit.

$$\underset{(a,\,b,\,c\ldots)}{[\overset{n}{C}]}=\Sigma\frac{\mathfrak{P}^{\,n}\overset{k}{C}}{1\ldots k},\ (k=1,\ 2\ \ldots.\ n)$$

$$(\overset{1}{S}_1,\ -\tfrac{1}{2}\overset{2}{S}_2,\ +\tfrac{1}{3}\overset{3}{S}_3\ \ldots.)$$

ex. gr. dentur: $a,\ b,\ c,\ d;$ sitque porro $n=3$

ponendo $k=1$, prodit $+\tfrac{1}{3}S_3$,
ponendo $k=2$, prodit $-\tfrac{1}{2}S_2 S_1$,
ponendo $k=3$, prodit $+\tfrac{1}{6}S_1 S_1 S_1$,

ideoque erit

$$abc+abd+acd+bcd=\tfrac{1}{3}(a^3+b^3+e^3+d^3)-\tfrac{1}{2}(a^2+b^2+c^2+d^2)(a+b+c+d)$$
$$+\tfrac{1}{6}(a+b+c+d)^3.$$

De evolvenda functione *Cos nx*.

§. 24.

Jam constat esse

$$2.\,Cos\,nx=(Cos\,x+Sin\,x\sqrt{-1})^n+(Cos\,x-Sin\,x\sqrt{-1})^n,$$

vel, statuendo $\sqrt{(1-Cos^2 x)}$ pro $Sin\,x$

$$2\,Cos\,n\,x=\{Cos\,x+\sqrt{(Cos^2 x-1)}\}^n+\{Cos\,x-\sqrt{(Cos^2 x-1)}\}^n.$$

Supra autem habuimus formulam

$$a^n+b^n=\alpha^n-n.\,\alpha^{n-2}\beta+\frac{n.\,n-3}{1.2}\alpha^{n-4}\beta^2\ \ldots.$$

cujus seriei terminus ultimus erit $\pm 2\beta^{\frac{n}{2}}$ vel $\pm n.\alpha.\beta^{\frac{n-1}{2}}$.

Statuamus nunc: $a=Cos\,x+\sqrt{(Cos^2 x-1)}$, $b=Cos\,x-\sqrt{(Cos^2 x-1)}$, ideo erit $a+b=2\,Cos\,x$, $ab=1$, et dehinc $\alpha=2\,Cos\,x$, $\beta=1$; quam ob rem habemus:

$$2.\,Cos\,n\,x=(2Cos\,x)^n-n(2Cos\,x)^{n-2}+\frac{n.n-3}{1.2}(2Cos\,x)^{n-4}-\frac{n.(n-4)(n-5)}{1.2.3}(2Cos\,x)^{n-6}\ldots$$

terminus ultimus erit ± 2, vel $\pm n\,(2\,Cos\,x)$, prout n numerus sit par vel impar.

Sicque ope laudatae formulae simplici via nacti sumus solutionem, quae vulgari hujus problematis tractandi methodo post plures demum ambages erui solet (Conf. Lacroix traité etc. Tome I. p. 263 sqq.).

Corrigenda.

Pag. 5.	lin. 18.	lege:	relatio
— ib.	— 26.	—	qui
— 10.	— 2.	—	$f'\overset{0}{a}$
— 11.	— 1.	—	$(i+k)^{n-1}$
— 12.	— 25.	—	illam vel per hanc.
— 19.	— 23.	—	${}^{m}\overset{1}{\mathfrak{B}}a^{m-1}$