Towards a general Doob-Meyer decomposition theorem

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Abstract

Both the Doob-Meyer and the Graversen-Rao decomposition theorems can be proved following an approach based on predictable compensators of discretizations and weak-L^1 technique, which was developed by K.M. Rao. It is shown that any decomposition obtained by Rao's method gives predictability of compensators without additional assumptions (like submartingality in the original Doob-Meyer theorem or finite energy in the Graversen-Rao theorem).

1 Introduction

In his seminal papers [11] and [12], P.A. Meyer proved that any submartingale belonging to so called class (D) admits a unique decomposition into a sum of a uniformly integrable martingale and a "natural" (nowadays: "predictable") integrable increasing process. More than twenty years later, S.E. Graversen and M. Rao [3] obtained a Doob-Meyer type decomposition for processes "with finite energy", in general without uniqueness. While the original Doob-Meyer theorem was motivated by needs of potential theory, and only later found interesting probabilistic applications (vide: stochastic integration), the latter result was used in analysis of Markov processes [3] and quite recently proved to be a useful tool in investigations of the structure of Dirichlet processes and their extensions [1].

Both the Doob-Meyer and the Graversen-Rao theorems can be proved following an approach based on predictable compensators of discretizations and weak-L^1 technique, which was developed by K.M. Rao [14]. In the present paper we show that any decomposition obtained by Rao's method leads to *predictable* compensators without additional assumptions (like submartingality in the original Doob-Meyer theorem or finite energy in the Graversen-Rao theorem).

The idea of the proof is in a sense similar to that from the paper [8] and is based on the celebrated Komlós theorem [10]. The details are however much more subtle and require other advanced tools, like limit theorems for stochastic integrals and tightness in so-called S-topology introduced in [7].

2 The result

Let $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a stochastic basis, satisfying the "usual" conditions, i.e. the filtration $\{\mathcal{F}_t\}$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}_T$. By

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convention, we set $\mathcal{F}_\infty = \mathcal{F}$. The family of stopping times with values in $[0,T]^* = [0,T] \cup \{+\infty\}$ and with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]^*}$ will be denoted by $\mathcal{T}$.

Let $\{X_t\}_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$, adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$ (i.e. for each $t \in [0,T]$, X_t is $\mathcal{F}_t$ measurable) and progressively measurable. We say that X is *of class* (D), if the family of random variables $\{X_\tau; \tau \in \mathcal{T}\}$ is uniformly integrable.

We say that X has càdlàg (or regular) trajectories, if its P-almost all trajectories are right-continuous and possess limits from the left on $[0,T]$.

For definitions of predictability, martingales etc. we refer to standard textbooks (e.g. [2], [4], [5], [9] or [13]).

Let $\theta_n = \{0 = t_0^n < t_1^n < t_2^n < \ldots < t_{k_n}^n = T\}$, $n = 1, 2, \ldots$, be condensing partitions of $[0,T]$, with

$$\max_{1\le k\le k_n} t_k^n - t_{k-1}^n \to 0, \quad \text{as } n \to \infty.$$

By "discretizations" $\{X_t^n\}_{t\in\theta_n}$ of $\{X_t\}_{t\in[0,T]}$ we mean the processes defined by

$$X_t^n = X_{t_k^n} \quad \text{if} \quad t_k^n \le t < t_{k+1}^n, \; X_T^n = X_T.$$

If random variables $\{X_t\}_{t\in[0,t]}$ are integrable, we can associate with any discretization X^n its "predictable compensator"

$$\begin{aligned} A_t^n &= 0 \quad \text{if} \quad 0 \le t < t_1^n, \\ A_t^n &= \sum_{j=1}^{k} E\big(X_{t_j^n} - X_{t_{j-1}^n} \big| \mathcal{F}_{t_{j-1}^n}\big) \quad \text{if} \quad t_k^n \le t < t_{k+1}^n, \; k = 1, 2, \ldots, k_n - 1, \\ A_T^n &= \sum_{j=1}^{k_n} E\big(X_{t_j^n} - X_{t_{j-1}^n} \big| \mathcal{F}_{t_{j-1}^n}\big). \end{aligned}$$

Notice that A_t^n is $\mathcal{F}_{t_{k-1}^n}$-measurable for $t_k^n \le t < t_{k+1}^n$, and so the processes A^n are predictable in a very intuitive manner, both in the discrete and in the continuous case. It is also clear, that for each $n \in \mathbb{N}$ the discrete-time process $\{M_t^n\}_{t\in\theta_n}$ given by

$$M_t^n = X_t^n - A_t^n, \quad t \in \theta_n,$$

is a martingale with respect to the discrete filtration $\{\mathcal{F}_t\}_{t\in\theta_n}$.

Theorem 1 *Let $\{X_t\}_{t\in[0,T]}$ be a càdlàg process* of class (D) *with respect to the stochastic basis $\mathcal{B}$.*

If for some condensing sequence $\{\theta_n\}_{n\in\mathbb{N}}$ the corresponding random variables $\{A_T^n\}_{n\in\mathbb{N}}$ are uniformly integrable, then one can find a uniformly integrable martingale $\{M_t\}_{t\in[0,T]}$ and a predictable integrable càdlàg process $\{A_t\}_{t\in[0,T]}$ of class (D) *such that we have the decomposition*

$$X_t = M_t + A_t, \quad t \in [0,T]. \tag{1}$$

An immediate consequence of predictability of $\{A_t\}$ is contained in the following

Corollary 2 *If $X_t = M_t' + A_t'$, $t \in [0,T]$, is another decomposition with properties described in Theorem 1, then $N_t = A_t - A_t'$, $t \in [0,T]$, is a uniformly integrable* continuous *martingale.*

It is clear that if we can attribute to $\{A_t\}$ some additional properties (e.g. it is non-decreasing or has finite variation or zero quadratic variation ...) then the martingale N in Corollary 2 must be zero and we obtain the uniqueness of the decomposition. In less standard cases this idea has been exploited in [3] (for Markov processes) and [1] (for weak Dirichlet processes).

One may ask what are the processes with "exploding" sequences of compensators, i.e. with $\{A_T^n\}_{n\in\mathbb{N}}$ not uniformly integrable. A variety of such processes can be constructed using the idea of self-cancellation of jumps, as in the following example.

Example 3 Let $\{r_n\}_{n\in\mathbb{N}}$ be a Rademacher sequence (i.i.d. with $P(r_n = 1) = P(r_n = -1) = 1/2$). Let $\{t_n\}_{n\in\mathbb{N}}$ be a (deterministic) sequence of times decreasing to 0. Define

$$X_t = \sum_n I_{[t_n,t_{n-1})}(t)\, r_n \frac{1}{\sqrt{n}}\ ,$$

and consider the natural filtration generated by X. Notice that X has regular trajectories, is adapted and bounded, hence of class (D).

We shall prove that X does not admit any decomposition of the form $X_t = M_t + A_t$, where M is a uniformly integrable martingale and A is a predictable, integrable càdlàg process. By contrary, suppose we are given such a representation. Then we have

$$\Delta M_{t_n} = \Delta X_{t_n} - \Delta A_{t_n} = r_n \frac{1}{\sqrt{n}} - r_{n+1}\frac{1}{\sqrt{n+1}} - \Delta A_{t_n}.$$

Using the facts that ΔM_{t_n} has null conditional expectation with respect to $\mathcal{F}_{t_n-}$ and ΔA_{t_n} is $\mathcal{F}_{t_n-}$- measurable, we obtain $\Delta A_{t_n} = -r_{n+1}\frac{1}{\sqrt{n+1}}$ and $\Delta M_{t_n} = r_n \frac{1}{\sqrt{n}}$. In particular, with probability one

$$\sum_n |\Delta M_{t_n}|^2 = +\infty,$$

what is impossible, since the quadratic variation of a martingale is finite.

3 A remark on the Graversen-Rao theorem

Following [3] we say that X is a process *of finite energy* if along a condensing sequence $\{\theta_n\}_{n\in\mathbb{N}}$ of partitions

$$\sup_n E\left[\sum_{t_i^n\in\theta_n} (X_{t_i^n} - X_{t_{i-1}^n})^2\right] < +\infty. \tag{2}$$

Of course, if X is of finite energy, $|X_t|^2$ is integrable for every $t \le T$ and $\sum_{s\le T}\Delta X_s^2$ is integrable, where $\Delta X_s = X_s - X_{s-}$. It is also easy to see that any process with finite energy satisfies one of the main assumptions of our Theorem 1: the sequence $\{A_T^n\}$ is bounded in L^2, hence uniformly integrable.

Further, it is not difficult to show that for each $\varepsilon > 0$ there exists a stopping time τ_ε such that $P(\tau_\varepsilon < T) < \varepsilon$ and

$$E \sup_{t\in[0.T]} |X_{\tau_\varepsilon\wedge t}|^2 < +\infty.$$

This property gives some kind of localization in class (D), but in general we do not know whether processes with finite energy form a subclass of class (D) processes.

Thus we are not able to show that the Graversen-Rao decomposition theorem is contained in our Theorem 1. Moreover, we have no examples showing that it is necessary to complete the assumptions of the Graversen-Rao theorem (e.g. by considering processes of class (D)).

What we want to stress is the fact that in the *sketch* of the proof given in [3] one can find convergence of quantities like

$$E\int_0^T A_t\, dC_t,$$

where C_t is an increasing *integrable, possibly unbounded* process. Corresponding limits are taken for granted, without paying any attention to details.

In the next section we rigourously perform similar computations and we find the class (D) property unavoidable.

4 Proof of Theorem 1

We will work with notation introduced in Section 2.

By the uniform integrability of $\{A_T^n\}$ we can find a subsequence $\{A_T^{n_k}\}$ convergent weakly in L^1 to some random variable α. This gives us the desired decomposition

$$X_t = M_t + A_t,$$

where

$$M_t = E(X_T - \alpha | \mathcal{F}_t)$$

is a uniformly integrable martingale (we take a càdlàg version of this process) and $A_t = X_t - M_t$ is a càdlàg process.

The essential novelty is contained in the proof of predictability of the process $\{A_t\}$, where we apply the Komlós theorem [10] in a similar way as it was done in [8], in the proof of the classical Doob-Meyer decomposition theorem, and then explore some properties of so-called S-topology introduced in [7].

Just as in the paper [8], we can find a further subsequence $\{n_{k_l}\}$ such that as $N \to \infty$

$$B_T^N = \frac{1}{N}\sum_{l=1}^{N} A_T^{n_{k_l}} \to \alpha = A_T, \qquad \text{a.s. and in } L^1. \tag{3}$$

It follows that, as $N \to \infty$,

$$M_T - \frac{1}{N}\sum_{l=1}^{N} M_T^{n_{k_l}} \to 0 \text{ a.s. and in } L^1, \tag{4}$$

where $M_T = X_T - A_T$ and $M_T^n = X_T^n - A_T^n = X_T - A_T^n$.

Next let us consider natural interpolations $\{\widetilde{M}_t^n\}_{t\in[0,T]}$ of the discrete-time martingales $\{M_t^n\}_{t\in\theta_n}$ to a uniformly integrable martingale with respect to the full filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. In other words,

$$\widetilde{M}_t^n = E\big(M_T^n | \mathcal{F}_t\big), \quad t \in [0,T].$$

It is a routine computation to verify that we have a decomposition

$$\widetilde{M}_t^n = \widetilde{X}_t^n - \widetilde{A}_t^n, \quad t \in [0,T],$$

where

$$\begin{array}{lll} \widetilde{X}_0^n = X_0, & \widetilde{X}_t^n = E\big(X_{t_k^n} \big| \mathcal{F}_t\big) & \text{if } \ t_{k-1}^n < t \le t_k^n, \ k = 1,2,\dots,k_n. \\ \widetilde{A}_0^n = 0, & \widetilde{A}_t^n = A_{t_k^n}^n & \text{if } \ t_{k-1}^n < t \le t_k^n, \ k = 1,2,\dots,k_n. \end{array}$$

The processes $\widetilde{A}^n$ are adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and their trajectories are *left continuous*, hence they are *predictable* by the very definition of the predictable σ-field.

Notice that for $t \in \theta_n$,

$$\widetilde{M}_t^n = M_t^n, \quad \widetilde{A}_t^n = A_t^n,$$

and, in particular,

$$\widetilde{M}_T^n = M_T^n, \quad \widetilde{A}_T^n = A_T^n.$$

We have also

Lemma 4 *The sequence $\{\widetilde{A}^n\}$ is uniformly of class (D), i.e. the family $\{\widetilde{A}_\tau^n : n \in \mathbb{N}, \tau \in \mathcal{T}\}$ is uniformly integrable.*

Proof. By the very definition

$$\widetilde{A}_\tau^n = \sum_{k=1}^{k_n} A_{t_k^n}^n I(t_{k-1}^n < \tau \le t_k^n).$$

Since τ is a stopping time, the event $\{t_{k-1}^n < \tau \le t_k^n\}$ belongs to $\mathcal{F}_{t_k^n}$. If we define

$$\rho^n(\tau) = 0 \quad \text{if } \tau = 0, \quad \rho^n(\tau)(\omega) = t_k^n \quad \text{if } t_{k-1}^n < \tau \le t_k^n, \tag{5}$$

then $\rho^n(\tau)$ is a stopping time with respect to the discrete filtration $\{\mathcal{F}_t\}_{t\in\theta_n}$, and

$$\widetilde{A}_\tau^n = A_{\rho^n(\tau)}^n.$$

By the discrete Doob-Meyer decomposition, $A_{\rho^n(\tau)}^n = X_{\rho^n(\tau)} - M_{\rho^n(\tau)}^n$, where X is of class (D) and $\{M^n\}$ is a sequence of discrete time martingales, with uniformly integrable terminal values $M_T^n = X_T - A_T^n$. □

Set

$$\widetilde{R}_t^N = \frac{1}{N} \sum_{l=1}^{N} \widetilde{M}_t^{n_{k_l}},$$

and observe that (4) implies uniform in probability convergence of martingales $\widetilde{R}^N$ to the martingale M.

$$P\big(\sup_{t\in[0,T]} |M_t - \widetilde{R}_t^N| > \varepsilon\big) \to 0, \ \text{as } N \to \infty, \ \varepsilon > 0. \tag{6}$$

Fix a stopping time $\tau \in \mathcal{T}$. Since $\{\widetilde{R}_\tau^N\}_{N\in\mathbb{N}}$ is uniformly integrable, we obtain from the above that

$$\widetilde{R}_\tau^N \to M_\tau \text{ in } L^1. \tag{7}$$

In what follows we shall suppress the subscript k_l in the subsequence n_{k_l}. With $\rho^n(t)$ defined by (5) we have $\widetilde{X}^n_t = E(X_{\rho^n(t)}|\mathcal{F}_t)$ and we can rewrite (7) in the form

$$\frac{1}{N}\sum_{m=1}^{N}\big(E(X_{\rho^m(\tau)}|\mathcal{F}_\tau) - \widetilde{A}^m_\tau\big) - \big(X_\tau - A_\tau\big) =$$
$$\frac{1}{N}\sum_{m=1}^{N} E(X_{\rho^m(\tau)} - X_\tau|\mathcal{F}_\tau) + \big(A_\tau - \frac{1}{N}\sum_{m=1}^{N}\widetilde{A}^m_\tau\big) \to 0, \text{ in } L^1 .$$

As m tends to infinity, $\rho^m(\tau) \searrow \tau$ and by the right continuity of $\{X_t\}$, $X_{\rho^m(\tau)} \to X_\tau$ a.s. Since $\{X_t\}$ is of class (D), the latter convergence holds also in L^1, hence $E(X_{\rho^m(\tau)} - X_\tau|\mathcal{F}_\tau) \to 0$, in L^1. Finally we obtain that for any stopping time τ

$$\widetilde{B}^N_\tau = \frac{1}{N}\sum_{m=1}^{N}\widetilde{A}^m_\tau \to A_\tau \text{ in } L^1. \tag{8}$$

This fact allows us to deduce a further remarkable property of the sequence $\{\widetilde{A}^m\}$.

Lemma 5 *For each stopping time $\tau \in \mathcal{T}$, $\widetilde{A}^m_\tau$ converges to A_τ weakly in L^1.*

Proof. Fix $\tau \in \mathcal{T}$ and suppose that for some bounded random variable Z and along some subsequence $\{m_r\}$

$$E\widetilde{A}^{m_r}_\tau Z \to c \neq EA_\tau Z. \tag{9}$$

Due to the "subsequence property" of the Komlós theorem, the relation (3) remains unchanged if we replace the subsequence $\{n_{k_l}\}$ with its subsequence $\{m_r\}$. Consequently also (6) and (8) hold, hence

$$\frac{1}{N}\sum_{r=1}^{N}\widetilde{A}^{m_r}_\tau \to A_\tau \text{ in } L^1.$$

This is in contradiction with (9). $\square$

Notice that $\widetilde{A}^n_\tau$'s are $\mathcal{F}_{\tau-}$ measurable and so by (8) the same property belongs to A_τ. We have thus checked one of the two conditions equivalent to the predictability of a càdlàg process (see e.g. [4], Theorem 4.33). The other condition requires that $A_\tau = A_{\tau-}$ a.s. on $\{\tau < +\infty\}$ for every totally inaccessible stoping time. We may and do assume that $A_\tau \geq A_{\tau-}$ a.s or $A_\tau \leq A_{\tau-}$ a.s on $\tau < +\infty$ (otherwise set e.g. $G = \{A_\tau \geq A_{\tau-}\} \in \mathcal{F}_\tau$, then $\tau' = \tau I_G + (+\infty)I_{G^c}$ is totally inaccessible and satisfies $A_{\tau'} \geq A_{\tau'-}$).

Let $\{C_t\}$ be a continuous nonnegative increasing process such that the process $\{I(\tau \leq t) - C_t\}_{t\in[0,T]}$ is a uniformly integrable martingale of zero mean. Since τ is totally inaccessible, $P(\tau = 0) = 0$ and we have $C_t = 0$ a.s.

Fix $K > 0$ and define stopping times

$$\eta_K = \inf\{t \in [0,T] : C_t > K\} \wedge T. \tag{10}$$

Notice that by continuity of C and $C_0 = 0$ a.s. we have $\eta_K > 0$ a.s.

We shall prove that

$$E\int_0^T \widetilde{A}^m_t\,dC_{t\wedge\eta_K}\to E\int_0^T A_t\,dC_{t\wedge\eta_K},\ \text{as } m\to\infty. \tag{11}$$

At first we shall ensure uniform integrability of the integrals.

Lemma 6 *Let $\{D_t\}_{t\in[0,T]}$ be a* bounded *increasing adapted continuous process, $D_0=0$ a.s. and let $\{B^i_t\}_{i\in\mathbb{I}}$ be a family of càdlàg or* càglàd *processes which are* uniformly of class (D). *Then the family of integrals $\{\int_0^T B^i_s\,dD_s\}_{i\in\mathbb{I}}$ is uniformly integrable.*

Proof.We may assume that $D_T\le 1$. Let $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ be a convex increasing function such that $\Phi(u)/u\to+\infty$ as $u\to+\infty$ and

$$\sup_{i,\tau} E\Phi(|B^i_\tau|)<+\infty.$$

Consider stopping times $\{\eta_{k/n}: k=1,2,\ldots,n, n\in\mathbb{N}\}$ defined for D by (10) and observe that

$$\begin{aligned}\sup_{i,n} E\Phi\big(|\sum_{k=1}^n B^i_{\eta_{k/n}}(D_{\eta_{k/n}}-D_{\eta_{(k-1)/n}})|\big) &\le \sup_{i,n} E\Phi\big(\sum_{k=1}^n |B^i_{\eta_{k/n}}|\frac{1}{n})\big)\\ &\le \sup_{i,n}\frac{1}{n}\sum_{k=1}^n E\Phi(|B^i_{\eta_{k/n}}|)<+\infty\end{aligned}$$

Since a.s. $\int_0^T B^i_s\,dD_s=\lim_{n\to\infty}\sum_{k=1}^n B^i_{\eta_{k/n}}(D_{\eta_{k/n}}-D_{\eta_{(k-1)/n}})$, the proof is complete. □

Corollary 7 *In fact we have proved uniform integrability of the larger family*

$$\big\{\int_0^T B^i_s\,dD_s : i\in\mathbb{I}\big\}\cup\big\{\sum_{k=1}^n B^i_{\eta_{k/n}}(D_{\eta_{k/n}}-D_{\eta_{(k-1)/n}}) : i\in\mathbb{I}, n\in\mathbb{N}\big\}.$$

Let us return to the proof of (11). Assume for brevity that $K\in\mathbb{N}$. Suppose that for some $\delta>0$ and along some subsequence $\{m'\}$

$$\big|E\int_0^T \widetilde{A}^{m'}_t\,dC_{t\wedge\eta_K}-E\int_0^T A_t\,dC_{t\wedge\eta_K}\big|>\delta. \tag{12}$$

We have by Corollary 7

$$E\int_0^T A_t\,dC_{t\wedge\eta_K}=\lim_{n\to\infty}\sum_{k=1}^{K\cdot n} EA_{\eta_{k/n}}(C_{\eta_{k/n}}-C_{\eta_{(k-1)/n}}).$$

On the other hand, by Lemma 5 for fixed $n\in\mathbb{N}, k\le K\cdot n$ we have

$$EA_{\eta_{k/n}}(C_{\eta_{k/n}}-C_{\eta_{(k-1)/n}})=\lim_{m'\to\infty} E\widetilde{A}^{m'}_{\eta_{k/n}}(C_{\eta_{k/n}}-C_{\eta_{(k-1)/n}}).$$

It follows that one can find a subsequence $m'_n \to \infty$ of $\{m'\}$ such that

$$
\begin{aligned}
E \int_0^T A_t \, dC_{t\wedge\eta_K} &= \lim_{n\to\infty} \sum_{k=1}^{K\cdot n} E\widetilde{A}^{m'_n}_{\eta_{k/n}} (C_{\eta_{k/n}} - C_{\eta_{(k-1)/n}}) \\
&= \lim_{n\to\infty} \int_0^T \pi_n(\widetilde{A}^{m'_n})_t \, d\pi_n(C)_t, \qquad (13)
\end{aligned}
$$

where $\pi_n(B)$ is the discretization of the process B at random times $0 \le \eta_{1/n} \le \eta_{2/n} \ldots \le \eta_K \le T$. We shall prove that along the subsequence $\{m'_n\}$

$$
E \int_0^T \widetilde{A}^{m'_n}_t \, dC_{t\wedge\eta_K} - E \int_0^T \pi_n(\widetilde{A}^{m'_n})_t \, d\pi_n(C)_t \to 0.
$$

Given (13), this will contradict (12).

In view of Corollary 7 it is enough to prove that for arbitrary sequence $\{m_n\}$

$$
\int_0^T \widetilde{A}^{m_n}_t \, dC_{t\wedge\eta_K} - \int_0^T \pi_n(\widetilde{A}^{m_n})_t \, d\pi_n(C)_t \to 0 \text{ in probability.}
$$

Let $\overline{A}^n$ be the càdlàg version of the process $\widetilde{A}^n$. Obviously we have $\int_0^T \widetilde{A}^{m_n}_t \, dC_{t\wedge\eta_K} = \int_0^T \overline{A}^{m_n}_{t-} \, dC_{t\wedge\eta_K}$. We have also for $k > 0$

$$
\pi_n(\widetilde{A}^{m_n})_{\eta_{k/n}} = \overline{A}^{m_n}_{\eta_{k/n}-}
$$

and so

$$
\int_0^T \pi_n(\widetilde{A}^{m_n})_t \, d\pi_n(C)_t = \int_0^T \overline{A}^{m_n}_{t-} \, d\pi_n(C)_t.
$$

We were thus able to reduce the problem to the convergence

$$
\int_0^T \overline{A}^{m_n}_{t-} \, d(C_{t\wedge\eta_K} - \pi_n(C)_t) \to 0 \text{ in probability.}
$$

We are going to apply results of [6]. In this context we need to recall the notion of S-tightness, i.e. uniform tightness in so-called S-topology on the Skorokhod space $D([0,T] : \mathbb{R}^1)$ introduced in [7] (see Proposition 3.1 there).

Let $\{X^\alpha\}$ be a family of stochastic processes with càdlàg trajectories on $[0,T]$. For a càdlàg function $x \in D([0,1] : \mathbb{R}^1)$ denote by $N_a^b(x)$ the number of up-crossings given levels $a < b$, $a, b \in \mathbb{R}^1$, on the interval $[0,T]$. Set also $\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|$. The family $\{X^\alpha\}$ is said to be S-tight if the family $\{\|X^\alpha\|_\infty\}$ is bounded in probability and for each pair $a < b$ of reals the family $\{N_a^b(X^\alpha)\}$ is bounded in probability.

Lemma 8 *The family $\{\overline{A}^n\}$ is S-tight.*

Proof. Any trajectory of $\overline{A}^n$ can be obtained by change of time of the corresponding trajectory of A^n (this change of time is related to the discretization θ_n and it eliminates the value 0 taken by A^n on $[0, t_1^n)$). Hence we have

$$
\|\overline{A}^n\|_\infty = \|A^n\|_\infty, \quad N_a^b(\overline{A}^n) \le N_a^b(A^n), \; a < b, a, b \in \mathbb{R}^1.
$$

To prove S-tightness of the family $\{A^n\}$ we observe first that due to the discrete nature of processes A^n it is sufficient to compute the quantities $\|A^n\|_\infty$ and $N_a^b(A^n)$ over the finite set θ_n. Further, on θ_n we have $A_t^n = X_t^n - M_t^n$, where $\{X^n\}$ is a restriction of the càdlàg process X and M^n is a martingale with respect to the discrete filtration $\{\mathcal{F}_t\}_{t\in\theta_n}$. Since $\sup_n E|M_T^n| \leq E|X_T| + \sup_n E|A_T^n| < +\infty$, we obtain S-tightness of $\{M^n\}$ by standard martingale inequalities. And S-tightness of the family $\{X^n\}$ of discretizations of a càdlàg process is obvious. □

Given S-tightness of the sequence $\{\overline{A}^n\}$ we are completely in the framework considered in [6]. We cannot however simply apply Theorem 3.11 of [7] and then Theorem 1 of [6] for we do not control the convergence of $\overline{A}_0^n = A_{t_1^n}^n$. Instead we can use Theorem 7 of [6] which states that any limit in distribution of our sequence of stochastic integrals is again a stochastic integral with respect to the limit of the sequence $C_{t\wedge\eta_K} - \pi_n(C)_t$ which is 0. We have proved (11).

If (11) is established, the rest of the proof is straight-forward. We have

$$\begin{aligned}
EA_{\tau-}I(\tau \leq \eta_K) &= E\int_0^T A_{t-}\, dI(\tau \leq t\wedge\eta_K) \\
&= E\int_0^T A_{t-}\, dC_{t\wedge\eta_K} \\
&= E\int_0^T A_t\, dC_{t\wedge\eta_K} \qquad \text{for } C \text{ is continuous} \\
&= \lim_{m\to\infty} E\int_0^T \widetilde{A}_t^m\, dC_{t\wedge\eta_K} \qquad \text{by (11)} \\
&= \lim_{m\to\infty} E\int_0^T \widetilde{A}_t^m\, dI(\tau \leq t\wedge\eta_K) \\
&= \lim_{m\to\infty} E\widetilde{A}_\tau^m I(\tau \leq \eta_K) \\
&= EA_\tau I(\tau \leq \eta_K) \qquad \text{by Lemma 5.}
\end{aligned}$$

We have assumed that $A_\tau \geq A_{\tau-}$ or $A_\tau \leq A_{\tau-}$, so we obtain

$$A_{\tau-}I(\tau \leq \eta_K) = A_\tau I(\tau \leq \eta_K) \text{ a.s.}, \; K > 0.$$

Since C_t is integrable, $P(\eta_K < T) \to 0$, as $K \to \infty$. Hence $A_\tau = A_{\tau-}$ a.s. and the theorem follows.

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