An eduction of the Lengevin equation

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Abstract

We discuss limiting procedures which support the interpretation of stochastic differential equations.

1 Introduction

When somebody writes down differential equations, a commonly accepted procedure is being used: the differential equation is the limit for difference equations built upon a well-understood model.

It is rather difficult to find a similar level of evidence in the area of stochastic differential equations. For example, let us consider so-called "Langevin equation", for the sake of brevity in dimension one only:

$$\frac{d^2X}{dt^2}(t) + \alpha\frac{dX}{dt}(t) = \sigma\frac{dB_t}{dt}. \tag{1}$$

In this equation $X(t)$ is the coordinate of the sample particle, α is a "viscosity coefficient" and dB_t/dt is "intended to represent the effect of a small shock at time t" (see [5]). While an explanation along the line above may be satisfactory for most of physicists, certainly it is not rigorous eduction of the Langevin equation. The approach of mathematicians is also typical. They simply do not bother about the process of eduction and pass immediately to the integral version:

$$\frac{dX}{dt}(t) = \frac{dX}{dt}(0) - \alpha(X(t) - X(0)) - \sigma B(t), \tag{2}$$

where $B(t)$ is a Brownian motion and X is as before ([5]). Sometimes it is also nice to introduce the velocity process $V(t) = dX/dt(t)$ and consider the Langevin equation in the most elegant form

$$V(t) = V(0) - \alpha\int_{]0,t]} V(s)\,ds - \sigma B(t). \tag{3}$$

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Even if an attempt to convince the reader is undertaken (as in the classical textbook by Breiman [1] or monograph [7]) some external and secret forces (the process of "momentum transfer") are invited and equipped with all necessary but not justified (e.g. continuity in time) properties.

In numerical approximations as well as in simulations one can use a discrete version of (3) (see e.g. [6]), but contrary to the deterministic case the discretization does not help in better understanding of (3): why do *random* forces act *regularly* in time?

The present note aims at providing a "naive" mechanical model in which viscosity and diffusion appear as macroscopic quantities and there is nothing artificial at the microscopic level.

2 The model

We shall deal with a rarefied simple gas and a sample particle of smoke with mass M. Suppose an elastic collision of the sample particle with a particle of gas happens. If the particle of smoke has the velocity V and the gas particle has mass m, $m < M$, and velocity U, then the change of V after collision will be

$$\Delta V = C(U - V), \tag{4}$$

where $C = C(M,m) = 2m/(M+m)$ (for simplicity we restrict ourselves to one dimensional space). Further, suppose such collisions take place at subsequent random times $\tau_1, \tau_2, \ldots$, with gas particles possessing the same masses but randomly chosen velocities $U_1, U_2, \ldots$. Then we have

$$\Delta V(\tau_k) = C(U_k - V(\tau_k -)), \tag{5}$$

where $U_1, U_2, \ldots$ are independent random variables with normal distribution $\mathcal{N}(0, kT/m)$. Here k is the Boltzmann constant and T is the absolute temperature. The variance of U_k must be equal to kT/m because of the "law of equipartition of energy" which determines the average kinetic energy per degree of freedom:

$$\frac{1}{2} m E(U_k)^2 = \frac{1}{2} kT. \tag{6}$$

Finally, suppose that random times $\tau_1, \tau_2, \ldots$ are such that the number of collisions up to time t is a Poisson process

$$N(t) = N^\lambda(t) = \sum_{k=1}^{\infty} \mathbb{I}_{[\tau_k, +\infty)}(t), \quad t \in \mathbb{R}^+,$$

with intensity $\lambda\, dt$. This gives us the independence of the number of collisions in disjoint time intervals. Let us notice that λ is *the average number of* *collisions* in unit time interval and that the average number of collisions in the given interval is proportional to the length of the interval.

Summing (5) up to time t we get

$$V(t) - V(0) = CU(t) - C\int_{]0,t]} V(s-)\,dN(s), \tag{7}$$

where $U(t) = U^{m,\lambda}(t) = \sum_{\tau_k \le t} U_k$. We can easily solve this equation:

$$V(t) = C\big(Y(t)\big)^{-1}\Big(\int_{]0,t]} Y(s)\,dU(s) + V(0)\Big), \tag{8}$$

where

$$Y(t) = Y^{m,\lambda}(t) = \exp(-(\ln(1 - C(M,m)))N^{\lambda}(t))$$

and we use the Lebesgue-Stieltjes integral.

Let us assume that $\lambda \to +\infty$ and $m \to 0$ in such a way that

$$2\lambda m \longrightarrow b, \quad 0 < b < +\infty. \tag{9}$$

Then

$$Y^{m,\lambda}(t) \underset{\mathcal{P}}{\longrightarrow} e^{\frac{b}{M}t}, \quad t \in I\!R^{+}, \tag{10}$$

and functionally

$$C(M,m)\,U^{m,\lambda}(t) \underset{\mathcal{D}}{\longrightarrow} \frac{\sqrt{2bkT}}{M}B(t), \tag{11}$$

where $\{B(t),\ t \in I\!R^{+}\}$ is a standard Brownian motion. Given (10) and (11), it is natural to expect that the solution (8) converges functionally to the process

$$V(t) = e^{-\frac{b}{M}t}\Big(\frac{\sqrt{2bkT}}{M}\int_{]0,t]} e^{\frac{b}{M}s}\,dB(s) + V(0)\Big), \tag{12}$$

which solves the Langevin equation

$$M(V(t) - V(0)) + b\int_{]0,t]} V(s)\,ds = \sqrt{2bkT}B(t). \tag{13}$$

Both conjectures are true. The point is this is not an *ad hoc* result but a corollary to the very general limit result for stochastic integrals - Theorem 2.6 in [2] or Theorem 1 in [8] (see also [3] for somewhat different approach and examples illustrating the theory).

We have obtained the solution of the Langevin equation for the particular model as the limit of solutions of simpler equations. The following are worth emphasizing:

- The coefficient b in (13) is a macroscopic quantity (interpreted as the resistance of the environment caused by the "viscosity" of the gas) and it does not appear at the microscopic level described by equation (7).

- Relation (9) provides a natural interpretation for the coefficient b. Similar considerations for more complicated models may contribute to the analysis of the "Einstein relation", as defined in [4].

- The final argument on passing to the limit was based on very general limit results and so seems to be applicable in many other, more advanced cases.

- The physical content of this paper is, of course, not very substantial. But it was not the intention of the paper to consider the most general case and to go into technicalities. Instead, we have tried to focus on the fact that the stochastic differential equation (7) is, in some sense, much better description of the model than the commonly accepted limiting equation (13).

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